Math 126, Section D - Spring 2014
Midterm II
May 20, 2014

Name: ______________________________

Student ID Number: ______________

Section: DA 11:30-12:20 by Hailun ☐ DB 12:30-1:20 by Hailun ☐
DC 11:30-12:20 by Bo Peter ☐ DD 12:30-1:20 by Bo Peter ☐

exercise	**possible**	**score**
1	14	
2	12	
3	12	
4	12	
total	50	

- Check that this booklet has all the exercises indicated above.
- TURN OFF YOUR CELL PHONE.
- Write your name and your student ID.
- This is a 50 minute test.
- You may use a scientific calculator and one 8.5×11 inch sheet of (two-sided) handwritten notes. All other electronic devices (including graphing calculators) are forbidden.
- Unless otherwise indicated, your answers should be exact instead of decimal approximations. For example $\frac{\pi}{4}$ is an exact answer and is preferable to its decimal approximation 0.78.
- Unless otherwise indicated, show your work and justify all your answers. Box your $\boxed{\text{final answer}}$.

$\boxed{\text{Sample solution}}$

Exercise 1 (5+2+2+5=14 points).

Consider the curve $\vec{r}(t) = ((t^2-2)^2, t^4, t^2)$.

$a)$ Compute $\vec{T}(t)$ for general t.

Solution: We have

$$\begin{aligned}\vec{r}'(t) &= (4(t^2-2)t, 4t^3, 2t) = (4t^3-8t, 4t^3, 2t)\\ |\vec{r}'(t)| &= \sqrt{32t^6-64t^4+68t^2} = 2t\sqrt{8t^4-16t^2+17}\\ \vec{T}(t) &= \frac{1}{\sqrt{32t^6-64t^4+68t^2}}(4t^3-8t, 8t^3, 2t)\end{aligned}$$

$b)$ Show that the curve lies in the plane $x-y+4z=4$.

Solution: Simply check that

$$(t^2-2)^2 - t^4 + 4t^2 = (t^4-4t^2+4) - t^4 + 4t^2 = 4$$

$c)$ Find one (non-zero) vector that is parallel to $\vec{B}(1)$.

Hint: Think about what $b)$ means for the osculating plane and for the position of the vectors $\vec{T}(t), \vec{N}(t), \vec{B}(t)$. You can use those insights to solve $c)$ and $d)$ with very little calculations.

Solution: The whole curve lies in the plane $x-y+4z=4$. Hence this must be the osculating plane for any t. Hence $\vec{B}(t)$ is parallel to $(1,-1,4)$ at any time t.

$d)$ Find one (non-zero) vector that is parallel to $\vec{N}(1)$.

Solution: We know that $\vec{T}(1), \vec{N}(1), \vec{B}(1)$ are pairwise orthogonal unit vectors. Hence any non-zero vector that is orthogonal to both $\vec{T}(1)$ and $\vec{B}(1)$ is parallel to $\vec{N}(1)$. Note that

$$\vec{r}'(1) = (4-8, 4, 2) = (-4, 4, 2)$$

(or $\vec{T}(1) = \frac{1}{\sqrt{36}}(-4,4,2) = \frac{1}{3}(-2,2,1)$). Hence, we can simply compute

$$\vec{r}'(1) \times (1,-1,4) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -4 & 4 & 2 \\ 1 & -1 & 4 \end{vmatrix} = (16+2, 16+2, 0) = (18, 18, 0)$$

Hence $\boxed{(1,1,0)}$ is parallel to $\vec{N}(1)$.

Alternative solution: one can also compute $\vec{r}''(t) = (12t^2-8, 12t^2, 2)$. Then $r'(1) = (-4,4,2)$ and $r''(1) = (4,12,2)$. Then $B(1)$ is $r'(1) \times r''(1) = (-16, 16, -64)$ normalized.

Exercise 2 (8+4=12 points).

Consider the surface in $\mathbb{R}^3$ that is defined by equation $2x^2+yz+y^3+xz^2=28$.
$a)$ Use implicit differentiation to compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Solution: We have

$$\frac{\partial}{\partial x}\left(2x^2+yz+y^3+xz^2\right)=4x+yz_x+z^2+2x\cdot z\cdot z_x \stackrel{!}{=} 0$$
$$\Rightarrow \quad z_x\cdot\left[y+2xz\right]=-4x-z^2 \quad \Rightarrow \quad \boxed{z_x=-\tfrac{4x+z^2}{y+2xz}}$$

and

$$\frac{\partial}{\partial y}\left(2x^2+yz+y^3+xz^2\right)=z+y\cdot z_y+3y^2+2x\cdot z\cdot z_y \stackrel{!}{=} 0$$
$$\Rightarrow \quad z_y\cdot\left[y+2xz\right]=-z-3y^2 \quad \Rightarrow \quad \boxed{z_y=-\tfrac{z+3y^2}{y+2xz}}$$

$b)$ Compute the tangent plane of the surface at $(2,2,2)$.

Solution: For $(2,2,2)$

$$z_x=-\frac{8+4}{2+8}=-\frac{6}{5} \quad \text{and} \quad z_y=-\frac{2+12}{2+8}=-\frac{7}{5}$$

and the equation of the tangent plane is

$$\boxed{T(x,y)=2-\frac{6}{5}(x-2)-\frac{7}{5}(y-2)}$$

Exercise 3 (12 points).

The equation $z^2 = 2x^2 + xy + y^2$ describes a surface in $\mathbb{R}^3$. Find all points on this surface that are closest to $(x_0, y_0, z_0) = (1, 2, 0)$?
Use the 2nd derivative test to show that the points you found are indeed the closest ones.

Solution: The squared distance of a point (x,y,z) to $(1,2,0)$ is

$$(x-1)^2 + (y-2)^2 + (z-0)^2$$

which we want to minimize over the surface. We know that each point on surface satisfies $z^2 = 2x^2 + xy + y^2$. Hence we need to minimize the function

$$h(x,y) = (x-1)^2 + (y-2)^2 + 2x^2 + xy + y^2 = 3x^2 + xy + 2y^2 - 2x - 4y + 5$$

Then

$$h_x(x,y) = 6x + y - 2 \qquad h_y(x,y) = x + 4y - 4$$

Both are 0 if

$$\left| \begin{array}{c} 6x + y - 2 = 0 \\ x + 4y - 4 = 0 \end{array} \right| \Rightarrow \left| \begin{array}{c} 6(4-4y) + y - 2 = 0 \\ x = 4 - 4y \end{array} \right| \Rightarrow \left| \begin{array}{c} 22 - 23y = 0 \\ x = 4(1-y) \end{array} \right| \Rightarrow x = \frac{4}{23}, y = \frac{22}{23}$$

Moreover

$$h_{xx} = 6 \qquad h_{xy} = 1 \qquad h_{yy} = 4$$

and

$$\begin{vmatrix} 6 & 1 \\ 1 & 4 \end{vmatrix} = 23 > 0$$

Hence this is indeed a minimum. The points are

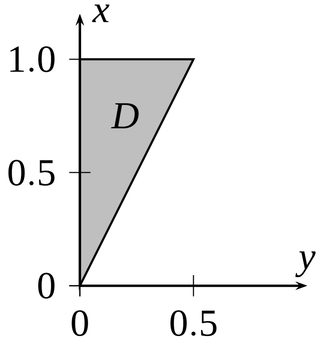
$$\boxed{\left(\frac{4}{23}, \frac{22}{23}, \sqrt{\frac{604}{529}}\right) \quad \text{and} \quad \left(\frac{4}{23}, \frac{22}{23}, -\sqrt{\frac{604}{529}}\right)}$$

Exercise 4 (12 points).

Evaluate the integral

$$\int_0^{1/2} \int_{2y}^{1} y \cdot \cos\left(\frac{\pi}{2}x^3\right)\, dx\, dy$$

Solution: The region of integration looks as follows



Hence we can switch the order of integration as

$$\begin{aligned}
\int_0^{1/2} \int_{2y}^{1} y \cdot \cos\left(\frac{\pi}{2}x^3\right)\, dxdy &= \int_0^1 \cos(\frac{\pi}{2}x^3)\Big(\int_0^{x/2} y\, dy\Big)dx \\
&= \int_0^1 \frac{1}{8}x^2 \cos\left(\frac{\pi}{2}x^3\right) dx \\
&= \frac{1}{8} \cdot \frac{2}{3\pi} \sin\left(\frac{\pi}{2}x^3\right)\Big|_0^1 = \frac{1}{12\pi}
\end{aligned}$$

using integration by substitution.