Math 126, Section D - Spring 2014
Midterm II
May 20, 2014

Name: ________________________________
Student ID Number: ________________

Exercise possible score

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>14</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>total</td>
<td>50</td>
<td></td>
</tr>
</tbody>
</table>

• Check that this booklet has all the exercises indicated above.

• TURN OFF YOUR CELL PHONE.

• Write your name and your student ID.

• This is a 50 minute test.

• You may use a scientific calculator and one 8.5 × 11 inch sheet of (two-sided) handwritten notes. All other electronic devices (including graphing calculators) are forbidden.

• Unless otherwise indicated, your answers should be exact instead of decimal approximations. For example \( \frac{\pi}{4} \) is an exact answer and is preferable to its decimal approximation 0.78.

• Unless otherwise indicated, show your work and justify all your answers. Box your final answer.

Sample solution
Exercise 1 (5+2+2+5=14 points).

Consider the curve \( \mathbf{r}(t) = ((t^2 - 2)^2, t^4, t^2) \).

\( a) \) Compute \( \mathbf{T}(t) \) for general \( t \).

**Solution:** We have

\[
\mathbf{r}'(t) = (4(t^2 - 2)t, 4t^3, 2t) = (4t^3 - 8t, 4t^3, 2t)
\]

\[
|\mathbf{r}'(t)| = \sqrt{32t^6 - 64t^4 + 68t^2} = 2t \sqrt{8t^4 - 16t^2 + 17}
\]

\[
\mathbf{T}(t) = \frac{1}{\sqrt{32t^6 - 64t^4 + 68t^2}}(4t^3 - 8t, 8t^3, 2t)
\]

\( b) \) Show that the curve lies in the plane \( x - y + 4z = 4 \).

**Solution:** Simply check that

\[(t^2 - 2)^2 - t^4 + 4t^2 = (t^4 - 4t^2 + 4) - t^4 + 4t^2 = 4\]

\( c) \) Find one (non-zero) vector that is parallel to \( \mathbf{B}(1) \).

**Hint:** Think about what \( b) \) means for the osculating plane and for the position of the vectors \( \mathbf{T}(t), \mathbf{N}(t), \mathbf{B}(t) \). You can use those insights to solve \( c) \) and \( d) \) with very little calculations.

**Solution:** The whole curve lies in the plane \( x - y + 4z = 4 \). Hence this must be the osculating plane for any \( t \). Hence \( \mathbf{B}(t) \) is parallel to \((1, -1, 4)\) at any time \( t \).

\( d) \) Find one (non-zero) vector that is parallel to \( \mathbf{N}(1) \).

**Solution:** We know that \( \mathbf{T}(1), \mathbf{N}(1), \mathbf{B}(1) \) are pairwise orthogonal unit vectors. Hence any non-zero vector that is orthogonal to both \( \mathbf{T}(1) \) and \( \mathbf{B}(1) \) is parallel to \( \mathbf{N}(1) \). Note that

\[
\mathbf{r}'(1) = (4 - 8, 4, 2) = (-4, 4, 2)
\]

(or \( \mathbf{T}(1) = \frac{1}{\sqrt{36}}(-4, 4, 2) = \frac{1}{3}(-2, 2, 1) \)). Hence, we can simply compute

\[
\mathbf{r}'(1) \times (1, -1, 4) = \begin{vmatrix}
\vec{i} & \vec{j} & \vec{k} \\
-4 & 4 & 2 \\
1 & -1 & 4
\end{vmatrix} = (16 + 2, 16 + 2, 0) = (18, 18, 0)
\]

Hence \([1, 1, 0]\) is parallel to \( \mathbf{N}(1) \).

**Alternative solution:** one can also compute \( \mathbf{r}''(t) = (12t^2 - 8, 12t^2, 2) \). Then \( \mathbf{r}'(1) = (-4, 4, 2) \) and \( \mathbf{r}''(1) = (4, 12, 2) \). Then \( \mathbf{B}(1) \) is \( \mathbf{r}'(1) \times \mathbf{r}''(1) = (-16, 16, -64) \) normalized.
Exercise 2 (8+4=12 points).
Consider the surface in $\mathbb{R}^3$ that is defined by equation $2x^2 + yz + y^3 + xz^2 = 28$.

a) Use implicit differentiation to compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Solution: We have

$$\frac{\partial}{\partial x} \left(2x^2 + yz + y^3 + xz^2\right) = 4x + yz_x + z^2 + 2x \cdot z \cdot z_x \neq 0$$

$$\Rightarrow z_x \cdot [y + 2xz] = -4x - z^2 \Rightarrow z_x = -\frac{4x + z^2}{y + 2xz}$$

and

$$\frac{\partial}{\partial y} \left(2x^2 + yz + y^3 + xz^2\right) = z + y \cdot z_y + 3y^2 + 2x \cdot z \cdot z_y \neq 0$$

$$\Rightarrow z_y \cdot [y + 2xz] = -z - 3y^2 \Rightarrow z_y = -\frac{z + 3y^2}{y + 2xz}$$

b) Compute the tangent plane of the surface at (2, 2, 2).

Solution: For (2, 2, 2)

$$z_x = -\frac{8 + 4}{2 + 8} = -\frac{6}{5} \quad \text{and} \quad z_y = -\frac{2 + 12}{2 + 8} = -\frac{7}{5}$$

and the equation of the tangent plane is

$$T(x,y) = 2 - \frac{6}{5}(x - 2) - \frac{7}{5}(y - 2)$$
Exercise 3 (12 points).

The equation $z^2 = 2x^2 + xy + y^2$ describes a surface in $\mathbb{R}^3$. Find all points on this surface that are closest to $(x_0, y_0, z_0) = (1, 2, 0)$? Use the 2nd derivative test to show that the points you found are indeed the closest ones.

Solution: The squared distance of a point $(x, y, z)$ to $(1, 2, 0)$ is

$$(x - 1)^2 + (y - 2)^2 + (z - 0)^2$$

which we want to minimize over the surface. We know that each point on surface satisfies $z^2 = 2x^2 + xy + y^2$. Hence we need to minimize the function

$$h(x, y) = (x - 1)^2 + (y - 2)^2 + 2x^2 + xy + y^2 = 3x^2 + xy + 2y^2 - 2x - 4y + 5$$

Then

$$h_x(x, y) = 6x + y - 2 \quad h_y(x, y) = x + 4y - 4$$

Both are 0 if

$$\begin{vmatrix} 6x + y - 2 = 0 \\ x + 4y - 4 = 0 \end{vmatrix} \Rightarrow \begin{vmatrix} 6(4 - 4y) + y - 2 = 0 \\ x = 4 - 4y \end{vmatrix} \Rightarrow \begin{vmatrix} 22 - 23y = 0 \\ x = 4(1 - y) \end{vmatrix} \Rightarrow x = \frac{4}{23}, y = \frac{22}{23}$$

Moreover

$$h_{xx} = 6 \quad h_{xy} = 1 \quad h_{yy} = 4$$

and

$$\begin{vmatrix} 6 & 1 \\ 1 & 4 \end{vmatrix} = 23 > 0$$

Hence this is indeed a minimum. The points are

$$\left(\frac{4}{23}, \frac{22}{23}, \sqrt{\frac{604}{529}}\right) \quad \text{and} \quad \left(\frac{4}{23}, \frac{22}{23}, -\sqrt{\frac{604}{529}}\right)$$
Exercise 4 (12 points).

Evaluate the integral

\[ \int_0^{1/2} \int_{2y}^{1} y \cdot \cos \left( \frac{\pi}{2} x^3 \right) \, dx \, dy \]

Solution: The region of integration looks as follows

Hence we can switch the order of integration as

\[
\int_0^{1/2} \int_{2y}^{1} y \cdot \cos \left( \frac{\pi}{2} x^3 \right) \, dx \, dy = \int_0^{1} \cos \left( \frac{\pi}{2} x^3 \right) \left( \int_0^{x/2} y \, dy \right) \, dx
\]

\[
= \int_0^{1} \frac{1}{8} x^2 \cos \left( \frac{\pi}{2} x^3 \right) \, dx
\]

\[
= \frac{1}{8} \cdot \frac{2}{3\pi} \sin \left( \frac{\pi}{2} \left( \frac{1}{2} \right)^3 \right) \bigg|_0^1 = \frac{1}{12\pi}
\]

using integration by substitution.