No books allowed. You may use a scientific calculator and one $8\frac{1}{2} \times 11$ sheet of notes.

- Do not share notes.

- In order to receive credit, you must show your work and explain your reasoning (except on the “short answer” questions).

- Place a box around **YOUR FINAL ANSWER** to each question.

- If you need more room, use the backs of the pages and indicate to the grader where to find your work.

- Raise your hand if you have a question or need more paper.

Don’t open the test until everyone has a copy and the start of the test is announced.

GOOD LUCK!
1. (10=3+3+4 points) All the parts of this problem concern the vector function \( \mathbf{r}(t) \) that satisfies the following conditions: the acceleration is \( \mathbf{a}(t) = 6t \mathbf{i} + \mathbf{j} - 12t^2 \mathbf{k} \) and the initial position and velocity are given by \( \mathbf{r}(0) = 2 \mathbf{k} \) and \( \mathbf{v}(0) = \mathbf{i} + \mathbf{j} \).

(a) Write an equation of the normal plane to the curve described by \( \mathbf{r}(t) \) at the point where \( t = 0 \).

\[
\text{Can take } \mathbf{r}'(0) = \mathbf{v}(0) = \langle 1, 1, 0 \rangle \text{ as a normal to this plane.}
\]

\[
\text{and } \mathbf{r}'(0) = \langle 0, 0, 2 \rangle \text{ as a pt on this plane.}
\]

Hence, eqn is \( 1(x-0) + 1(y-0) + 0(z-2) = 0 \) or \( \sqrt{x+y} = 0 \)

(b) Compute the normal component of acceleration at \( t = 0 \).

\[
\mathbf{a}_N(0) = \left| \frac{\mathbf{r}'(0) \times \mathbf{r}''(0)}{|\mathbf{r}'(0)|} \right| = \frac{|\mathbf{v}(0) \times \mathbf{a}(0)|}{|\mathbf{v}(0)|}
\]

\[
\mathbf{v}(0) = \langle 1, 1, 0 \rangle \quad \Rightarrow \quad \mathbf{v}' \times \mathbf{a}' = \langle 0, 1, 1 \rangle
\]

\[
\mathbf{v}'(0) \times \mathbf{a}'(0) = 1
\]

\[
|\mathbf{v}'(0)| = \sqrt{2}
\]

\[
\Rightarrow \quad \mathbf{a}_N = \frac{1}{\sqrt{2}}
\]

(c) Find this vector function \( \mathbf{r}(t) \).

\[
\mathbf{a}(t) = \langle 6t, 1, -12t^2 \rangle \quad \Rightarrow \quad \mathbf{v}(t) = \langle 3t^2 + C_1, t + C_2, -4t^3 + C_3 \rangle
\]

\[
\mathbf{v}(0) = \langle 1, 0 \rangle \quad \Rightarrow \quad C_1 = 1, C_2 = 1, C_3 = 0
\]

\[
\Rightarrow \quad \mathbf{v}(t) = \langle 3t^2 + 1, t + 1, -4t^3 \rangle
\]

\[
\Rightarrow \quad \mathbf{r}(t) = \langle t^3 + t + K_1, \frac{t^2}{2} + t + K_2, -t^4 + K_3 \rangle
\]

\[
\mathbf{r}(0) = \langle 0, 0, 2 \rangle \quad \Rightarrow \quad K_1 = K_2 = 0, K_3 = 2
\]

\[
\Rightarrow \quad \mathbf{r}(t) = \langle t^3 + t, \frac{t^2}{2} + t, 2 - t^4 \rangle
\]
2. (8 points) The level curves of a function $f(x,y)$ are shown below.

![Level Curves](attachment:image.png)

Determine whether the following partial derivatives are positive (> 0), negative (< 0) or zero (= 0). No explanation of answers needed for this problem. Be sure to explain your answers on other problems!

(a) $\frac{\partial f}{\partial x}$ at the point $P$ is

(b) $\frac{\partial f}{\partial y}$ at the point $P$ is

(c) $\frac{\partial f}{\partial x}$ at the point $Q$ is

(d) $\frac{\partial f}{\partial y}$ at the point $Q$ is

Circle one:

- $> 0$
- $< 0$
- $= 0$
3. (6 = 2 + 2 + 2 points) Consider the function \( f(x, y) = e^{3x+5y-1} \).

(a) Calculate the partial derivatives \( f_x \) and \( f_y \).

\[
\begin{align*}
    f_x &= 3e^{3x+5y-1} \\
    f_y &= 5e^{3x+5y-1}
\end{align*}
\]

(b) Write an equation for the tangent plane to the graph of \( f(x, y) \) at the point \((2, -1, 1)\).

\[
1 = 2e^{3\cdot2+5(-1)+1} \quad \checkmark
\]

\( \Rightarrow \) this pt is on the graph

\[
\begin{align*}
    f_x(2, -1) &= 3, \\
    f_y(2, -1) &= 5
\end{align*}
\]

\[
2 = 3(x-2) + 5(y+1) + 1
\]

(c) Use the linear approximation for \( f \) at \((2, -1)\) to estimate the value \( f(1.8, -0.9) \).

\[
\begin{align*}
    f(1.8, -0.9) &\approx 3(1.8-2) + 5(-0.9+1) + 1 \\
    &= -0.6 + 0.5 + 1 = 0.9
\end{align*}
\]
4. (10 points) Find the local maximum and minimum values and saddle points of the function \( f(x, y) = (x^2 + y)e^{y/2} \).

Critical points:

\[
\begin{align*}
\frac{\partial f}{\partial x} &= 2x \cdot e^{y/2} \\
\frac{\partial f}{\partial y} &= e^{y/2} + \frac{1}{2}(x^2 + y) \cdot e^{y/2} = \frac{1}{2} e^{y/2} (2 + x^2 + y) = 0
\end{align*}
\]

\[
\begin{align*}
x &= 0 & \Rightarrow & & y = 0 \\
2 + x^2 + y &= 0 & \Rightarrow & & y = -2
\end{align*}
\]

\[
\Rightarrow (0, -2) \text{ is the only critical point}
\]

\[
\begin{align*}
\frac{\partial^2 f}{\partial x^2} &= 2e^{y/2} \\
\frac{\partial^2 f}{\partial x \partial y} &= 2xe^{y/2} = xe^{y/2} \\
\frac{\partial^2 f}{\partial y^2} &= \frac{1}{4} e^{y/2} (2 + x^2 + y) + \frac{1}{2} \cdot e^{y/2} \\
&= \frac{1}{4} e^{y/2} (4 + x^2 + y)
\end{align*}
\]

\[
\begin{align*}
D(0, -2) &= \begin{vmatrix} 2e^{-1} & 0 \\ 0 & \frac{1}{4} e^{-1} \cdot 2 \end{vmatrix} = e^{-2} > 0 \\
\text{and} & \quad f_{xx}(0, -2) = 2e^{-1} > 0
\end{align*}
\]

\[
\Rightarrow \text{have a local min.}
\]

The value of this local min is

\[
f(0, -2) = -2 \cdot e^{-1} = -\frac{2}{e}
\]
5. (8 points)

(a) Evaluate the integral

\[
\int_0^1 \int_0^1 \sqrt{x^3+1} \, dx \, dy.
\]

\[
= \int_0^1 \int_0^1 \sqrt{x^3+1} \, dy \, dx
\]

\[
= \int_0^1 \left( \sqrt{x^3+1} \right) dy \bigg|_{y=0}^{y=1}
\]

\[
= \int_0^1 x^2 \sqrt{x^3+1} \, dx
\]

\[
= \frac{1}{3} \left( \frac{2}{3} \right) \left( \frac{2}{3} \right) = \frac{4}{27} \left( \frac{2}{3} \right) = \frac{8}{81}
\]

(b) Compute the volume of the solid bounded by the paraboloids \( z = x^2 + y^2 \) from below and \( z = \frac{x^2}{2} + \frac{y^2}{2} + 1 \) from above. (Hint: draw a picture.)

Integration:

\[
\frac{x^2 + y^2}{2} + 1 = x^2 + y^2
\]

\[
\Rightarrow \frac{x^2 + y^2}{2} = 1 \Rightarrow \frac{x^2 + y^2}{2} = \frac{2}{2}
\]

\[
V = \iiint_D \left( \frac{x^2 + y^2}{2} + 1 \right) - (x^2 + y^2) \, dV
\]

\[
= \int_0^1 \int_0^1 \int_0^{\sqrt{2}} (1 - \frac{r^2}{2}) \, r \, dr \, d\theta
\]

\[
= 2\pi \int_0^1 \left( \frac{1}{2} - \frac{r^2}{2} \right) \, dr
\]

\[
= 2\pi \left[ \frac{1}{2} r - \frac{r^3}{6} \right]_0^1
\]

\[
= 2\pi \left[ \frac{1}{2} \cdot 1 - \frac{1}{6} \right] = 2\pi \cdot \frac{1}{3} = \frac{2\pi}{3}
\]