1. (13 pts)

(a) At time $t = 0$ a particle is passing through the $xy$-plane at the location $(0, 0, 0)$ with velocity $\mathbf{v}(0) = \langle -1, -2, -6 \rangle$. The acceleration of the particle is given by $\mathbf{a}(t) = \langle \pi \sin(\pi t), 0, 4 \rangle$. Find the speed of the particle at the next time it passes through the $xy$-plane. (Hint: First find the position function).

$$\vec{v}(t) = \int \vec{a}(t)\,dt = \langle -\cos(\pi t) + c_1, c_2, 4t + c_3 \rangle$$

$$\vec{v}(0) = \langle -1, -2, -6 \rangle \Rightarrow -1 + c_1 = -1, \quad c_2 = -2, \quad c_3 = -6$$

$$\vec{v}(t) = \langle -\cos(\pi t), -2, 4t - 6 \rangle$$

$$\vec{r}(t) = \int \vec{v}(t)\,dt = \langle -\tfrac{1}{\pi}\sin(\pi t) + d_1, -2t + d_2, 2t^2 - 6t + d_3 \rangle$$

$$\vec{r}(0) = \langle 0, 0, 0 \rangle \Rightarrow d_1 = 0, d_2 = 0, d_3 = 0$$

$$\vec{r}(t) = \langle -\tfrac{1}{\pi}\sin(\pi t), -2t, 2t^2 - 6t \rangle$$

CROSS XY-PLANE $\Rightarrow 2t^2 - 6t = 0$

$\Rightarrow 2t(t-3) = 0 \qquad t = 0$ or $t = 3$

$$\text{SPEED} = |\vec{v}(3)| = \sqrt{(-(-1))^2 + (-2)^2 + (12-6)^2}$$

$$= \boxed{\sqrt{41}}$$

(b) Consider the surface given implicitly by $z^2 = e^{(x^2y^3-1)} + \dfrac{3x}{y} + \ln(y)$. Find the equation for the tangent plane at $(x, y, z) = (1, 1, -2)$.

$$2z\frac{\partial z}{\partial x} = 2xy^3 e^{(x^2y^3-1)} + \frac{3}{y} \Rightarrow \frac{\partial z}{\partial x} = \frac{2e^0 + 3}{2(-2)} = -\frac{5}{4}$$

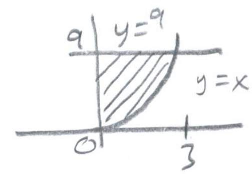
$$2z\frac{\partial z}{\partial y} = 3x^2y^2 e^{(x^2y^3-1)} - \frac{3x}{y^2} + \frac{1}{y} \Rightarrow \frac{\partial z}{\partial y} = \frac{3e^0 - 3 + 1}{2(-2)} = -\frac{1}{4}$$

$$\boxed{z + 2 = -\tfrac{5}{4}(x-1) - \tfrac{1}{4}(y-1)}$$

2. (14 pts)

(a) Reverse the order of integration and evaluate $\int_0^3 \int_{x^2}^9 xe^{y^2}\, dydx$.

$0 \le x \le 3,\ x^2 \le y \le 9 \implies$




$, x = \sqrt{y}$

$0 \le y \le 9,\ 0 \le x \le \sqrt{y}$

$\int_0^9 \int_0^{\sqrt{y}} x e^{y^2}\, dx\, dy$

$\int_0^9 \frac{1}{2} x^2 e^{y^2} \Big|_0^{\sqrt{y}}\, dy$

$\frac{1}{2} \int_0^9 y\, e^{y^2}\, dy$

$u = y^2$
$du = 2y\, dy$
$dy = \frac{1}{2y} du$

$\frac{1}{2} \int_0^{81} \frac{1}{2} e^u\, du$

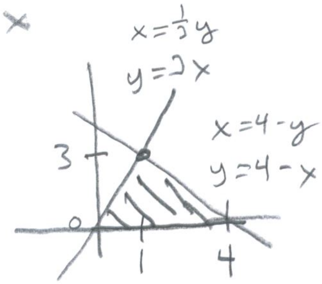
$\frac{1}{4} e^u \Big|_0^{81} = \boxed{\frac{1}{4}(e^{81} - 1)}$

(b) Find the volume of the solid bounded by $z = y^2$, $z = 0$, $y = 3x$ and $y = 4 - x$.

$\iint_D y^2\, dA = ?$

$\boxed{D}$ : Given $y = 3x,\ y = 4 - x$

ALSO $\underset{\uparrow \atop 0}{z} = y^2 \Rightarrow y = 0$




$\int_0^3 \int_{\frac{1}{3}y}^{4-y} y^2\, dx\, dy$ or $\int_0^1 \int_0^{3x} y^2\, dy\, dx + \int_1^4 \int_0^{4-x} y^2\, dy\, dx$

$\int_0^3 y^2 x \Big|_{\frac{1}{3}y}^{4-y}\, dy$

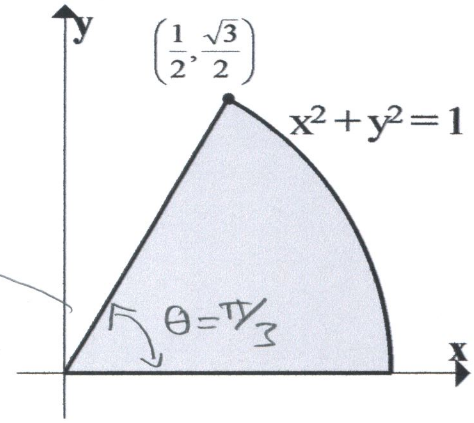
$\int_0^3 y^2(4-y) - y^2 \frac{1}{3}y\, dy$

$\int_0^3 4y^2 - y^3 - \frac{1}{3}y^3\, dy$

$\int_0^3 4y^2 - \frac{4}{3}y^3\, dy = \frac{4}{3}y^3 - \frac{1}{3}y^4 \Big|_0^3$

$= \frac{4}{3}(3)^3 - \frac{1}{3}(3)^4 = 36 - 27 = \boxed{9}$

3. (9 pts) Find the average value of $T(x, y) = 16x^2$ over the region $D$ shown below.




$$\text{AVE. VALUE} = \frac{1}{\text{AREA}} \iint_D 16x^2 \, dA$$

$$0 \le \theta \le \pi/3$$
$$0 \le r \le 1$$

$$\text{AREA} = \frac{1}{2}\theta r^2 = \frac{1}{2}\frac{\pi}{3}(1)^2 = \pi/6$$

ALSO CAN BE FOUND USING $\iint_D 1 \, dA$

$$\text{AVE. VALUE} = \frac{6}{\pi}\int_0^{\pi/3}\int_0^1 16r^2\cos^2\theta \; r\,dr\,d\theta$$

$$= \frac{6}{\pi}\int_0^{\pi/3} 16\cos^2\theta \; \frac{1}{4}r^4\Big|_0^1 \, d\theta$$

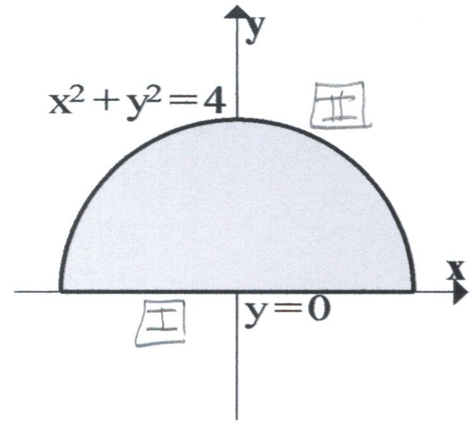
$$= \frac{24}{\pi}\int_0^{\pi/3} \cos^2\theta \, d\theta$$

$$= \frac{24}{\pi}\int_0^{\pi/3} \frac{1}{2}(1+\cos(2\theta))\,d\theta$$

$$= \frac{12}{\pi}\left(\theta + \frac{1}{2}\sin(2\theta)\Big|_0^{\pi/3}\right)$$

$$\boxed{= \frac{12}{\pi}\left(\frac{\pi}{3} + \frac{1}{2}\frac{\sqrt{3}}{2}\right) = 4 + \frac{3\sqrt{3}}{\pi}}$$

4. (14 pts) Find the absolute maximum and minimum values of $f(x, y) = y(x^2 + y^2) - 2y^2 + 1$ over the region $D$ shown below.




$f(x,y) = yx^2 + y^3 - 2y^2 + 1$

(i) $f_x = 2yx \overset{?}{=} 0 \Rightarrow x = 0$ OR $y = 0$

(ii) $f_y = x^2 + 3y^2 - 4y \overset{?}{=} 0$

$x = 0 \Rightarrow 3y^2 - 4y = 0 \Rightarrow y(3y-4) = 0$

$y = 0$ or $y = 4/3$

$y = 0 \Rightarrow x^2 = 0 \Rightarrow x = 0$

$\boxed{\text{CRITICAL POINTS } (0,0) \text{ AND } (0, 4/3)}$

I $y = 0 \Rightarrow z = f(x,0) = 1$ = constant height

II $x^2 + y^2 = 4 \Rightarrow z = 4y - 2y^2 + 1$

$z' = 4 - 4y \overset{?}{=} 0$

$\Rightarrow y = 1$

$\Rightarrow x^2 + 1 = 4$

$x = \pm\sqrt{3}$

$\boxed{\text{CRITICAL NUMBERS ON BOUNDARY } (-\sqrt{3}, 1), (\sqrt{3}, 1)}$

COMPARING HEIGHTS

$f(x,0) = 1$

$f(0, 4/3) = \frac{4}{3}\left(0 + \left(\frac{4}{3}\right)^2\right) - 2\left(\frac{4}{3}\right)^2 + 1 = \frac{64}{27} - \frac{32}{9} + 1 = \frac{64 - 96 + 27}{27} = \boxed{-\frac{5}{27} \text{ ABS. MIN}}$

$f(\pm\sqrt{3}, 1) = (1)(3+1) - 2(1)^2 + 1 = \boxed{3 \quad \text{ABS MAX}}$