Evaluate the following double integrals.

(a) (8 points) \( \iint_R x \sec^2(xy) \, dA, \quad R = [0, \pi/4] \times [0, 1] \)

\[
\int_0^{\pi/4} \int_0^1 x \sec^2(xy) \, dy \, dx = \int_0^{\pi/4} \tan(xy) \bigg|_{y=0}^1 \, dx \\
= \int_0^{\pi/4} \tan(x) \, dx \\
= \ln(\sec(x)) \bigg|_0^{\pi/4} \\
= \ln \sqrt{2}
\]

(b) (8 points) \( \iint_D 2xy \, dA \quad D \) is the triangle with vertices (0, 0), (1, 2) and (0, 3).

\[
\int_0^1 \int_2^{3-x} 2xy \, dy \, dx = \int_0^1 x y^2 \bigg|_2^{3-x} \, dx \\
= \int_0^1 9x - 6x^2 - 3x^3 \, dx \\
= \left[ \frac{9}{2} x^2 - 2x^3 - \frac{3}{4} x^4 \right]_0^1 \\
= \frac{7}{4}
\]
Find the absolute maximum of the function $f(x, y) = x + 2y - xy$ on the closed rectangular region with vertices (0,0), (0,2), (3,0) and (3,2).

First find the critical points.

$f_x(x, y) = 1 - y = 0$ gives $y = 1$

$f_y(x, y) = 2 - x = 0$ gives $x = 2$. 

There is one critical point at $(2, 1)$. $f(2, 1) = 2$. 

Now find the maximum of $f(x, y)$ on the boundary. There are four parts to this.

I. $x = 0$, $0 \leq y \leq 2$
   
   $f(0, y) = 2y$ is increasing so the maximum is $f(0, 2) = 4$.

II. $y = 0$, $0 \leq x \leq 3$
    
    $f(x, 0) = x$ is increasing so the maximum is $f(3, 0) = 3$.

III. $x = 3$, $0 \leq y \leq 2$
     
     $f(3, y) = 3 - y$ is decreasing so the maximum is $f(3, 0) = 3$.

IV. $y = 2$, $0 \leq x \leq 3$
    
    $f(x, 2) = 4 - x$ is decreasing so the maximum is $f(0, 2) = 4$.

Thus the maximum value of the function on this rectangle is $f(0, 2) = 4$. 
If three resistors with resistances $R_1$, $R_2$ and $R_3$ are connected in parallel, then the total resistance $R$ of the circuit is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

Suppose that the resistances are measured in ohms with $R_1 = 25$, $R_2 = 40$ and $R_3 = 50$, and that there is a possible error of 0.5 ohms in each measurement. Use differentials to estimate the maximum error in the calculated value of $R$.

$$\frac{1}{R} = \frac{1}{25} + \frac{1}{40} + \frac{1}{50} \quad \text{so} \quad R = \frac{200}{17}.$$ 

$$\frac{\partial R}{\partial R_1} = \frac{-1}{R^2} \cdot \frac{\partial R}{\partial R_1} = \frac{1}{R_1^2} \quad \text{so} \quad \frac{\partial R}{\partial R_1} = \frac{64}{289}.$$ 

$$\frac{\partial R}{\partial R_2} = \frac{-1}{R^2} \cdot \frac{\partial R}{\partial R_2} = \frac{1}{R_2^2} \quad \text{so} \quad \frac{\partial R}{\partial R_2} = \frac{25}{289}.$$ 

$$\frac{\partial R}{\partial R_3} = \frac{-1}{R^2} \cdot \frac{\partial R}{\partial R_3} = \frac{1}{R_3^2} \quad \text{so} \quad \frac{\partial R}{\partial R_3} = \frac{16}{289}.$$ 

$$dR = \frac{\partial R}{\partial R_1} \cdot dR_1 + \frac{\partial R}{\partial R_2} \cdot dR_2 + \frac{\partial R}{\partial R_3} \cdot dR_3 = \frac{64}{289} \cdot 0.5 + \frac{25}{289} \cdot 0.5 + \frac{16}{289} \cdot 0.5 = \frac{578}{578} \approx 0.18.$$

Find all the points on the curve $r = 1 + \cos \theta$ where the tangent line is horizontal.

We wish to find the points where \( \frac{dy}{d\theta} = 0 \)

$$y = r \sin \theta = (1 + \cos \theta)(\sin \theta) = \sin \theta + \cos \theta \sin \theta$$

$$\frac{dx}{d\theta} = \cos \theta - \sin^2 \theta + \cos^2 \theta$$

$$= \cos \theta - (1 - \cos^2 \theta) + \cos^2 \theta$$

$$= 2 \cos^2 \theta + \cos \theta - 1$$

$$= (2 \cos \theta - 1)(\cos \theta + 1)$$

Thus $\cos \theta = -1, \frac{1}{2}$ and $\theta = \pi, \pm \frac{\pi}{3}$.

In polar coordinates, the points are $(0, \pi)$, $(3/2, \pi/3)$ and $(3/2, -\pi/3)$.
Let \( r(t) = 3t^2 \mathbf{i} + t^3 \mathbf{j} + 3t^2 \mathbf{k} \). Find all times \( t \) when the normal component of acceleration is equal to 8.

We must solve \( \frac{|r' \times r''|}{|r'|} = 8 \) or \( |r' \times r''| = 8 |r'| \) for \( t \).

\[
\begin{align*}
r'(t) &= \langle 6t, 3t^3, 6t \rangle \\
r''(t) &= \langle 6, 6t, 6 \rangle \\
r' \times r'' &= \langle -18t^2, 0, 18t^2 \rangle = 18t^2 \langle -1, 0, 1 \rangle \\
|r' \times r''| &= 18t^2 \sqrt{2} \\
|r'(t)| &= \sqrt{(6t)^2 + (3t^2)^2 + (6t)^2} = 3t\sqrt{t^2 + 8} \\
18t^2 \sqrt{2} &= 8 \cdot 3t\sqrt{t^2 + 8} \quad (\text{note that at } t = 0 \text{ we have } a_N = 0.) \\
3\sqrt{2} t &= 4 \sqrt{t^2 + 8} \\
18t^2 &= 16(t^2 + 8) \\
t^2 &= 64
\end{align*}
\]

The solutions are \( t = -8, 8 \).