1. (a) (6 pts) Use the linear approximation to $g(x,y) = \dfrac{\sqrt{x^3+1}}{2y} + e^{xy}$ at $(0,1)$ to approximate the value of $g(0.1, 0.9)$.

$g_x(x,y) = \dfrac{3x^2}{4y\sqrt{x^3+1}} + ye^{xy} \Rightarrow g_x(0,1) = 0 + 1 = 1$

$g_y(x,y) = -\dfrac{\sqrt{x^3+1}}{2y^2} + xe^{xy} \Rightarrow g_y(0,1) = -\frac{1}{2} + 0 = -\frac{1}{2}$

$z_0 = g(0,1) = \frac{1}{2} + 1 = 3/2$

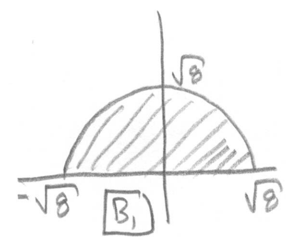
$g(0.1, 0.9) \approx L(0.1, 0.9) = \frac{3}{2} + 1 \cdot (0.1 - 0) - \frac{1}{2}(0.9 - 1)$

$= 1.5 + 0.1 + 0.05 = 1.65 = \frac{33}{20}$

(b) (8 pts) Consider the surface $f(x,y) = xy^2 + x + 2$. Find the absolute maximum and minimum over the region $R = \{(x,y) \mid y \geq 0, x^2 + y^2 \leq 8\}$.

CRITICAL PTS: $f_x(x,y) = y^2 + 1 \overset{?}{=} 0$ NEVER

no critical pts.



$\boxed{B_1}$ $y = 0$, $-\sqrt{8} \leq x \leq \sqrt{8} \Rightarrow z = f(x,0) = x + 2$ ← line

MAX/MIN ON B_1: $f(-\sqrt{8}, 0) = 2 - \sqrt{8} \approx -0.83$

$f(\sqrt{8}, 0) = 2 + \sqrt{8} \approx 4.83$

$\boxed{B_2}$ $y = \sqrt{8 - x^2}$, $-\sqrt{8} \leq x \leq \sqrt{8}$

$z = f(x, \sqrt{8-x^2}) = x(8-x^2) + x + 2 = 9x - x^3 + 2$

$\frac{dz}{dx} = 9 - 3x^2 = 0 \Rightarrow x^2 = 3$, $x = \pm\sqrt{3}$

$y = \sqrt{8-3} = \sqrt{5}$

$f(-\sqrt{3}, \sqrt{5}) = -\sqrt{3}\,5 - \sqrt{3} + 2$

$= 2 - 6\sqrt{3} \approx -8.39$

$f(\sqrt{3}, \sqrt{5}) = \sqrt{3}\,5 + \sqrt{3} + 2 = 2 + 6\sqrt{3} \approx 12.39$

ABS. MAX $= 2 + 6\sqrt{3}$

ABS. MIN $= 2 - 6\sqrt{3}$

10 $\quad x^2y - 9y - xy^2 + y^3$

2. (0 pts) Let $f(x, y) =$ [redacted]. Find and classify all critical points of $f(x, y)$. (Classify using appropriate partial derivative tests).

① $f_x(x,y) = 2xy - y^2 \overset{?}{=} 0 \Rightarrow \quad y(2x-y) = 0$

$y = 0 \quad \text{or} \quad y = 2x$

② $f_y(x,y) = x^2 - 9 - 2xy + 3y^2 \overset{?}{=} 0$

$\cdot\ y = 0 \Rightarrow x^2 - 9 \overset{?}{=} 0 \Rightarrow x^2 = 9 \qquad x = \pm 3$

$\cdot\ y = 2x \Rightarrow x^2 - 9 - 4x^2 + 12x^2 \overset{?}{=} 0$

$9x^2 = 9 \Rightarrow x^2 = 1 \qquad x = \pm 1$

$y = \pm 2$

Four critical points: $\boxed{(-3,0),\ (3,0),\ (-1,-2),\ (1,2)}$

Second Derivative Test

$f_{xx}(x,y) = 2y \quad , \quad f_{yy}(x,y) = -2x + 6y \quad , \quad f_{xy}(x,y) = 2x - 2y$

$D(x,y) = 2y(-2x+6y) - (2x-2y)^2$

$(-3,0) \Rightarrow D(-3,0) = 0 - (-6)^2 = -36 < 0$ SADDLE POINT

$(3,0) \Rightarrow D(3,0) = 0 - (6)^2 = -36 < 0$ SADDLEPOINT

$(-1,-2) \Rightarrow D(-1,-2) = -4(2-12) - (-2+4)^2 = 40 - 4 = 36 > 0$

$f_{xx}(-1,-2) = -4 < 0$ LOCALMAX

$(1,2) \Rightarrow D(1,2) = 4(-2+12) - (2-4)^2 = 40 - 4 = 36 > 0$

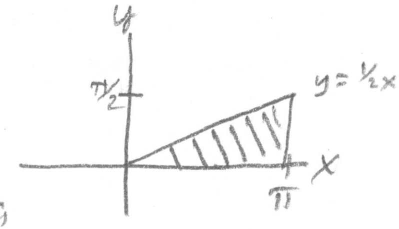
$f_{xx}(1,2) = 4 > 0$ LOCAL MIN

3. (a) (8 pts) Evaluate: $\int_0^{\frac{\pi}{2}} \int_{2y}^{\pi} \sin(x^2)\, dx dy$.

$0 \le y \le \pi/2$

$2y \le x \le \pi$

$y = \frac{1}{2}x$



REVERSING

$0 \le x \le \pi$

$0 \le y \le \frac{1}{2}x$

$$\int_0^{\pi} \int_0^{x/2} \sin(x^2)\, dy\, dx$$

$$\int_0^{\pi} \sin(x^2)\, y \Big|_0^{x/2} dx$$

$$\int_0^{\pi} \sin(x^2) \frac{x}{2}\, dx$$

$u = x^2$

$du = 2x\,dx$

$dx = \frac{1}{2x} du$

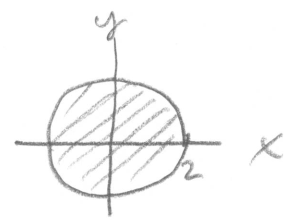
$$\int_0^{\pi^2} \sin(u) \frac{1}{4}\, du$$

$$-\frac{1}{4}\cos(u) \Big|_0^{\pi^2} = -\frac{1}{4}\cos(\pi^2) - {}^{-}\tfrac{1}{4} = \boxed{\tfrac{1}{4} - \tfrac{1}{4}\cos(\pi^2)}$$

(b) (8 pts) Set up and evaluate a double integral in polar coordinates in order to find the volume of the solid that is within the cylinder $x^2 + y^2 = 4$, above the plane $z = 1$, and below the surface $z + y - x^2 - y^2 = 3$.

$$z = 3 - y + x^2 + y^2$$

(NOTE: Always above $z = 1$ in this region)



$$\iint_D 3 - y + x^2 + y^2\, dA - \underbrace{\iint_D 1\, dA}_{\pi(2)^2}$$

$$\int_0^{2\pi}\int_0^2 (3 - r\sin(\theta) + r^2)\, r\,dr\,d\theta$$

$$\int_0^{2\pi}\int_0^2 3r - r^2\sin(\theta) + r^3\, dr\,d\theta = \int_0^{2\pi} \frac{3}{2}r^2 - \frac{1}{3}r^3\sin(\theta) + \frac{1}{4}r^4 \Big|_0^2 d\theta$$

$$= \int_0^{2\pi} \frac{3}{2}(2)^2 - \frac{1}{3}(2)^3\sin(\theta) + \frac{1}{4}(2)^4\, d\theta$$

$$= \int_0^{2\pi} 6 - \frac{8}{3}\sin(\theta) + 4\, d\theta$$

$$= 10\theta + \frac{8}{3}\cos(\theta) \Big|_0^{2\pi}$$

$$= (20\pi + \tfrac{8}{3}) - (0 + \tfrac{8}{3}) = 20\pi$$

$$\text{VOLUME} = 20\pi - 4\pi = \boxed{16\pi}$$

4. (10 pts) You are sitting at the origin with a water balloon of mass 1/2 kg and a water balloon cannon. Your math instructor is sitting on the xy-plane at the point $(0, 50, 0)$, in meters. The water balloon cannon always fires with an initial (vertical) z-component of velocity of 4 m/s.

There is a steady wind that blows with a constant force of $\langle 3, 0, 0\rangle$ Newtons. Assume the force from the wind and the force due to gravity (with $g = 9.8$ m/s^2) are the only forces acting on the water balloon when it is in the air.

At what initial velocity vector should you fire the cannon, so that the water balloon will land on your instructor? (That is, what are the x and y components of the initial velocity vector).

$$\vec{F} = \langle 3,0,0\rangle + \langle 0,0,-mg\rangle = \langle 3,0,-mg\rangle$$

$$\vec{a}(t) = \frac{1}{m}\vec{F} = \frac{1}{1/2}\langle 3,0,-\tfrac{1}{2}g\rangle = \langle 6,0,-g\rangle$$

$$\vec{v}(t) = \int \vec{a}(t)\,dt = \langle 6t+v_x, v_y, -gt+\underbrace{v_z}_{4}\rangle$$

$$\vec{r}(t) = \int \vec{v}(t)\,dt = \langle 3t^2+v_xt+\underbrace{c_1}_{0}, v_yt+\underbrace{c_2}_{0}, -\tfrac{g}{2}t^2+4t+\underbrace{c_3}_{0}\rangle$$

LAND ON INSTRUCTOR $\Rightarrow$ THE FOLLOWING EQUATIONS ARE SOLVABLE FOR t:

① $3t^2+v_xt = 0 \Rightarrow t(3t+v_x)=0$

② $v_y t = 50$

③ $-\frac{g}{2}t^2+4t = 0 \Rightarrow t(-\frac{g}{2}t+4)=0$

③ $\Rightarrow$ $-\frac{g}{2}t+4=0$, SO $t = 8/g$

② $\Rightarrow$ $v_y = \frac{50}{8/g} = \frac{50g}{8}$

① $\Rightarrow$ $3t+v_x=0$ $\quad v_x = -3(8/g) = -\frac{24}{g}$

$$\boxed{\vec{v}(0) = \langle -\tfrac{24}{g}, \tfrac{50g}{8}, 4\rangle}$$