MATH 126 Spring 2010 Exam 2

[1] $\vec{r}(t) = \langle t^2-3t, \frac{1}{\pi}\cos(\pi t), \frac{1}{\pi}\sin(\pi t)\rangle$

(a)(i) $\vec{r}\,'(t) = \langle 2t-3, -\sin(\pi t), \cos(\pi t)\rangle$

$\vec{r}\,''(t) = \langle 2, -\pi\cos(\pi t), -\pi\sin(\pi t)\rangle$

$$\begin{array}{ccc|ccc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} & \vec{k} \\ -3 & 0 & 1 & -3 & 0 & 1 \\ 2 & -\pi & 0 & 2 & -\pi & 0 \end{array}$$

$\vec{r}\,'(0) = \langle -3, 0, 1\rangle$

$\vec{r}\,''(0) = \langle 2, -\pi, 0\rangle$

$$K(0) = \frac{|\vec{r}\,'(0)\times\vec{r}\,''(0)|}{|\vec{r}\,'(0)|^3} = \frac{|\langle \pi, 2, 3\pi\rangle|}{|\langle -3, 0, 1\rangle|^3}$$

$$= \sqrt{\pi^2+4+9\pi^2} \Big/ \left(\sqrt{(-3)^2+0^2+1^2}\right)^3$$

$$= \sqrt{4+10\pi^2} \Big/ 10^{3/2} = \frac{1}{10}\frac{\sqrt{4+10\pi^2}}{\sqrt{10}} = \frac{1}{10}\sqrt{\frac{4+10\pi^2}{10}}$$

$$\approx 0.32046$$

(ii) $\text{speed} = v(t) = |\vec{v}(t)| = \sqrt{(2t-3)^2+\sin^2(\pi t)+\cos^2(\pi t)}$

$= \sqrt{(2t-3)^2+1}$

$v(0) = |\vec{v}(0)| = |\vec{r}\,'(0)| = \boxed{\sqrt{10}}$

(iii) $a_N = K(0)\,(v(0))^2 = \frac{1}{10}\sqrt{\frac{4+10\pi^2}{10}}\cdot(\sqrt{10})^2$

$= \sqrt{\frac{4+10\pi^2}{10}} = \sqrt{0.4+\pi^2}$

(b) $x=-2 \Leftrightarrow t^2-3t=-2 \Leftrightarrow t^2-3t+2=0$

$(t-1)(t-2)=0$

$t=1$, $t=2$

first time

$\vec{r}(1) = \langle -2, -\frac{1}{\pi}, 0\rangle$

$\vec{r}\,'(1) = \langle -1, 0, -1\rangle$

$\langle -1, 0, -1\rangle\cdot(\langle x,y,z\rangle - \langle -2, -\frac{1}{\pi}, 0\rangle) = 0$

$-(x+2) - z = 0$

$-x-2-z=0$

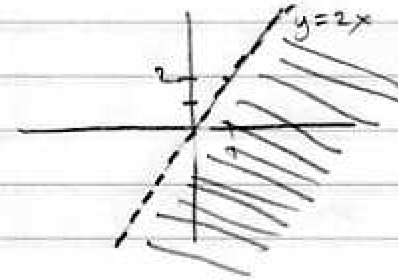
$z=-x-2$

$\boxed{2}$ (a) $x\ln(2x-y) + x\cos(y^2)$

is only defined for $2x - y > 0$

$\boxed{2x > y}$

$y = 2x$



(b) $f_x(x,y) = \ln(2x-y) + \frac{2x}{2x-y} + \cos(y^2)$

$f_y(x,y) = -\frac{x}{2x-y} - 2yx\sin(y^2)$

$f_x(\tfrac{1}{2},0) = \ln(2(\tfrac{1}{2})-0) + \frac{2(\frac{1}{2})}{2(\frac{1}{2})-0} + \cos(0^2) = 0+1+1 = 2$

$f_y(\tfrac{1}{2},0) = \frac{-\frac{1}{2}}{2(\frac{1}{2})-0} - 2(0)(\tfrac{1}{2})\sin(0^2) = -\frac{\frac{1}{2}}{1} = -\tfrac{1}{2}$

$z_0 = f(\tfrac{1}{2},0) = \tfrac{1}{2}\ln(2(\tfrac{1}{2})-0) + \tfrac{1}{2}\cos(0^2) = \tfrac{1}{2}$

$z - \tfrac{1}{2} = 2(x-\tfrac{1}{2}) - \tfrac{1}{2}(y-0)$

$z = \tfrac{1}{2} + 2x - 1 - \tfrac{1}{2}y$

$z = 2x - \tfrac{1}{2}y - \tfrac{1}{2}$

(c) $f(0.51, -0.01) \approx \tfrac{1}{2} + 2(0.51-0.5) - \tfrac{1}{2}(-0.01-0)$

$= \tfrac{1}{2} + 0.02 + 0.005$

$= \boxed{0.525}$

3

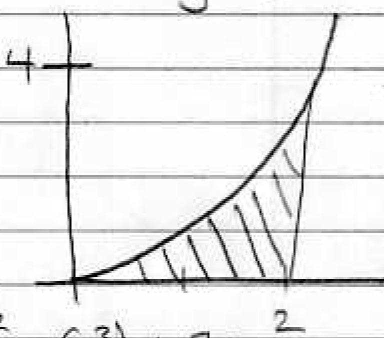
$0 \le y \le 4$

$\sqrt{y} \le x \le 2$

can also be described as

$0 \le x \le 2$

$0 \le y \le x^2$



$$\int_0^4 \underbrace{\int_{\sqrt{y}}^2 e^{(x^3)}\,dx}_{\text{NOT "DOABLE"}}\,dy = \int_0^2 \Big[\underbrace{\int_0^{x^2} e^{(x^3)}\,dy}_{\text{IS DOABLE}}\Big]dx$$

$$\int_0^2 e^{(x^3)} \int_0^{x^2} dy\ dx$$

$$\int_0^2 e^{(x^3)}\ y\Big|_0^{x^2}\ dx$$

$$\int_0^2 e^{(x^3)}\ x^2\ dx$$

$u = x^3$

$du = 3x^2\,dx$

$dx = \frac{1}{3x^2}\,du$

$x=0 \Rightarrow u=0$

$x=2 \Rightarrow u=8$

$$\frac{1}{3}\int_0^8 e^u\,du$$

$$\frac{1}{3}\ e^u\Big|_0^8$$

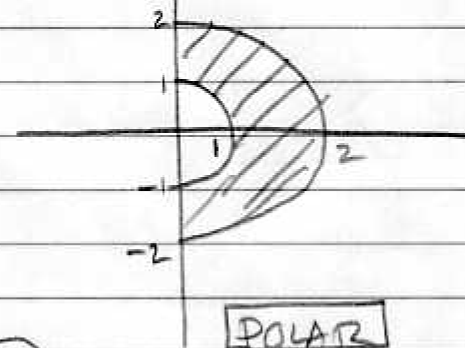
$$\boxed{\frac{1}{3}\left(e^8 - 1\right)}$$

$\boxed{4}$ $x^2+y^2+z^2=9$

$z^2 = 9 - x^2 - y^2$

$z = \pm\sqrt{9-x^2-y^2}$

upper hemisphere



$\boxed{\text{POLAR}}$

$-\pi/2 \le \theta \le \pi/2$

$1 \le r \le 2$

$$\iint_R \sqrt{9-x^2-y^2}\, dA$$

$$\int_{-\pi/2}^{\pi/2}\int_1^2 \sqrt{9-r^2}\; r\,dr\,d\theta$$

$$-\frac{1}{2}\int_{-\pi/2}^{\pi/2}\int_8^5 \sqrt{u}\, du\, d\theta = \frac{1}{2}\int_{-\pi/2}^{\pi/2}\int_5^8 \sqrt{u}\,du\,d\theta$$

$u = 9 - r^2$

$du = -2r\,dr$

$dr = \frac{1}{-2r}\,du$

$r=1 \Leftrightarrow u=8$

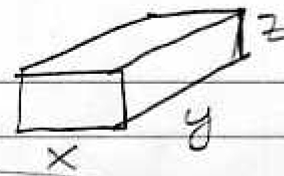
$r=2 \Leftrightarrow u=5$

$$\frac{1}{2}\int_{-\pi/2}^{\pi/2}\left[\frac{2}{3}u^{3/2}\Big|_5^8\right] d\theta$$

$$\frac{1}{3}\int_{-\pi/2}^{\pi/2}\left(8^{3/2}-5^{3/2}\right) d\theta$$

$$\frac{1}{3}\left(8^{3/2}-5^{3/2}\right)\underbrace{\theta\Big|_{-\pi/2}^{\pi/2}}_{=\pi}$$

$$= \boxed{\frac{\pi\left(8^{3/2}-5^{3/2}\right)}{3}}$$



5 $\text{Volume} = xyz = 9$

$\Rightarrow \quad z = 9/xy$

$\text{Cost} = 8xy + 1.5(2xz) + 1.5(2yz)$

$= 8xy + 3xz + 3yz$

$$C(x,y) = 8xy + \frac{27}{y} + \frac{27}{x}$$

(i) $C_x(x,y) = 8y - \frac{27}{x^2} \stackrel{?}{=} 0 \Rightarrow 8y = \frac{27}{x^2} \Rightarrow y = \frac{27}{8x^2}$

(ii) $C_y(x,y) = 8x - \frac{27}{y^2} = 0 \Rightarrow 8x = 27/y^2$

(i) & (ii) $\Rightarrow 8x = \frac{27}{(27/(8x^2))^2} \Rightarrow 8x = \frac{8^2}{27}x^4$

$\Rightarrow \cancel{x = 0}$ or $x^3 = \frac{27}{8} \Rightarrow \boxed{x = 3/2}$

ASIDE: $x=0$ OR $y=0$ GIVE CRITICAL POINTS AS WELL BECAUSE C_x or C_y ARE UNDEFINED, BUT THESE DON'T MAKE SENSE FOR THIS PROBLEM (volume wouldn't be 9).

$x = 3/2 \Rightarrow y = \frac{27}{8x^2} = \frac{27}{8(\frac{3}{2})^2} = \frac{3}{2}$ as well

$z = 9/xy = \frac{9}{(3/2)^2} = 4$

$\boxed{(x,y,z) = (\tfrac{3}{2}, \tfrac{3}{2}, 4)} \longleftarrow$ Dimensions

2nd derivative test

$C_{xx} = 54/x^3 \Rightarrow C_{xx}(\tfrac{3}{2}, \tfrac{3}{2}) = \frac{54}{(27/8)} = 16$

$C_{yy} = 54/y^3 \Rightarrow C_{yy}(\tfrac{3}{2}, \tfrac{3}{2}) = 16$

$C_{xy} = 8 \Rightarrow C_{xy}(\tfrac{3}{2}, \tfrac{3}{2}) = 8$

Thus, $D = 16 \times 16 - 8^2 = 192 > 0$

AND $C_{xx}(\tfrac{3}{2}, 3/2) = 16 > 0$

hence $(x,y) = (\tfrac{3}{2}, \tfrac{3}{2})$ is a local minimum of cost.