Solutions to Math 126CD, Au 2016, MT2

1. (10 points) The vector function $\mathbf{r}(t) = \langle 2t, 3t^2, t^3 \rangle$ sketches a curve in space. Compute the following at the point where $t = 1$.

(a) The equation of the normal plane.

$\vec{r}\,'(t) = \langle 2, 6t, 3t^2 \rangle \qquad \vec{r}\,'(1) = \langle 2, 6, 3 \rangle$

$\vec{r}(1) = \langle 2, 3, 1 \rangle$

Normal plane

$$2(x-2) + 6(y-3) + 3(z-1) = 0$$
$$2x + 6y + 3z = 25$$

(b) The curvature κ.

$\vec{r}\,''(t) = \langle 0, 6, 6t \rangle \qquad \vec{r}\,''(1) = \langle 0, 6, 6 \rangle$

$$\vec{r}\,' \times \vec{r}\,'' = 6 \begin{vmatrix} i & j & k \\ 2 & 6 & 3 \\ 0 & 1 & 1 \end{vmatrix} = 6\left((6-3)\vec{i} - (2-0)\vec{j} + (2-0)\vec{k}\right)$$
$$= \langle 18, -12, 12 \rangle = 6\langle 3, -2, 2 \rangle$$

$\|\vec{r}\,' \times \vec{r}\,''\| = 6\sqrt{9+4+4} = 6\sqrt{17}$

$\|\vec{r}\,'\| = \sqrt{4+36+9} = 7$

$$\kappa = \frac{6\sqrt{17}}{7^3}$$

(c) The unit normal vector $\mathbf{N}$.

$\|\vec{r}\,'(t)\| = \sqrt{4 + 36t^2 + 9t^4}$

$\vec{T}(t) = (4 + 36t^2 + 9t^4)^{-1/2} \langle 2, 6t, 3t^2 \rangle$

$\vec{T}\,'(t) = -\frac{1}{2}(4+36t^2+9t^4)^{-3/2}(72t + 36t^3)\langle 2, 6t, 3t^2 \rangle$

$\qquad + (4+36t^2+9t^4)^{-1/2} \langle 0, 6, 6t \rangle$

$\vec{T}\,'(1) = -\frac{1}{2} \cdot 7^{-3} \cdot 108 \langle 2, 6, 3 \rangle + 7^{-1} \langle 0, 6, 6 \rangle$

$\qquad = \frac{6}{7^3}\left[\langle -18, -54, -27 \rangle + \langle 0, 49, 49 \rangle\right] = \frac{6}{7^3}\langle -18, -5, 22 \rangle$

$\|\vec{T}\,'\| = \frac{6}{7^3}\sqrt{18^2 + 25 + 484} = \frac{6}{7^3}\sqrt{833} \rightarrow \vec{N}(1) = \frac{1}{\sqrt{833}}\langle -18, -5, 22 \rangle$

2. (8 points) A surface is given by the equation $\cos^2(x-2y)+y^2=2e^z+z$.

(a) Use implicit differentiation to compute the partial derivatives z_x and z_y.

$D_x: \; -2\cos(x-2y)\sin(x-2y)=(2e^z+1)z_x$

$$z_x=\frac{-2\cos(x-2y)\sin(x-2y)}{2e^z+1}$$

$D_y: \; 4\cos(x-2y)\sin(x-2y)+2y=(2e^z+1)z_y$

$$z_y=\frac{4\cos(x-2y)\sin(x-2y)+2y}{2e^z+1}$$

(b) Write down the equation of the tangent plane to this surface at the point $(2,1,0)$.

when $x=2$ $y=1$ $z=0$, $z_x=0$ $z_y=\frac{2}{3}$

$$z-0=0(x-2)+\frac{2}{3}(y-1)$$

$$z=\frac{2}{3}(y-1)$$

(c) Use linear approximation to approximate the value of z when $x=1.95$ and $y=1.01$.

$$z\approx\frac{2}{3}(1.01-1)=\frac{0.02}{3}\approx 0.0067$$

3. (12 points) Find and classify all critical points of the function

$$f(x,y) = 2xy + \frac{15}{4}x + \frac{1}{y} + \frac{1}{8}\ln x.$$

CPs

$$f_x = 2y + \frac{15}{4} + \frac{1}{8x} = 0$$

$$f_y = 2x - \frac{1}{y^2} = 0 \rightarrow 2x = \frac{1}{y^2} \rightarrow \frac{1}{x} = 2y^2$$

so, $0 = 2y + \frac{15}{4} + \frac{2y^2}{8} = 0$ or $y^2 + 8y + 15 = 0$

$(y+3)(y+5) = 0$

$y = -3 \quad y = -5$

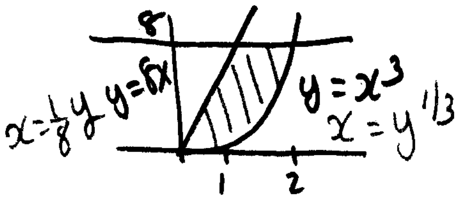
$x = \frac{1}{18}, \quad x = \frac{1}{50}$

classifying $\quad f_{xx} = \frac{-1}{8x^2} \quad f_{xy} = 2 \quad f_{yy} = \frac{2}{y^3} \quad D$

$(\frac{1}{18}, -3)$	$\frac{-\cancel{18^2}\ \cancel{81}\ 3}{\cancel{8}\ 4}$	2	$\frac{-\cancel{2}}{\cancel{27}}$	$+3-4<0$	saddle
$(\frac{1}{50}, -5)$	$\frac{-\cancel{50^2}\ \cancel{25}\ 5\ \cancel{2}\ <0}{\cancel{8}\ 4}$	2	$\frac{-\cancel{2}}{\cancel{125}\ 8}$	$5-4>0$	max

4. (10 points) D is the region in the first quadrant which is above the curve $y = x^3$, to the right of the line $y = 8x$ and below the horizontal line $y = 8$.

(a) Sketch the region D.



(b) Set up the integral $\int\int_D (2x)dA$ integrating with respect to x first. You may have to split the integral into two parts. Evaluate the integral(s).

$$\int_0^8\int_{\frac{1}{8}y}^{y^{1/3}} 2x\,dx\,dy = \int_0^8 x^2\Big|_{\frac{1}{8}y}^{y^{1/3}} dy = \int_0^8 y^{2/3} - \frac{1}{64}y^2\,dy$$

$$= \frac{3}{5}y^{5/3} - \frac{1}{64}\frac{y^3}{3}\Big|_0^8 = \frac{3\cdot 32}{5} - \frac{8}{3} = \frac{288-40}{15} = \frac{248}{15}$$

(c) Set up the integral $\int\int_D (2x)dA$ integrating with respect to y first. You may have to split the integral into two parts. Evaluate the integral(s).

$$\int_0^1\int_{x^3}^{8x} 2x\,dy\,dx + \int_1^2\int_{x^3}^{8} 2x\,dy\,dx$$

$$= \int_0^1 2xy\Big|_{x^3}^{8x} dx + \int_1^2 2xy\Big|_{x^3}^{8} dx$$

$$= \int_0^1 16x^2 - 2x^4\,dx + \int_1^2 16x - 2x^4\,dx = \frac{16x^3}{3} - \frac{2x^5}{5}\Big|_0^1 + 8x^2 - \frac{2x^5}{5}\Big|_1^2$$

$$= \frac{16}{3} - \cancel{\frac{2}{5}} + \left(32 - \frac{64}{5}\right) - \left(8 - \cancel{\frac{2}{5}}\right) = \frac{16}{3} + 24 - \frac{64}{5} = \frac{80+360-192}{15} = \frac{248}{15}$$