1. **[8 points]** An ant is standing on the surface $z = x^3 - 3xy + e^{xy}$ at the point $(1, 0)$.

(a) **[4 points]** If the ant were to walk East (that is, in the positive x direction), would it move up or down? Explain your reasoning.

$$\frac{\partial z}{\partial x} = 3x^2 - 3y + y\,e^{xy}$$

$$\left.\frac{\partial z}{\partial x}\right|_{(1,0)} = 3 > 0$$

Since the slope in the x-direction is positive the ant would move up.

(b) **[4 points]** Use differentials to estimate the ant's change in altitude when the ant travels from $(1, 0)$ to $(0.95, 0.12)$.

$$\frac{\partial z}{\partial x} = 3x^2 - 3y + y\,e^{xy} \Rightarrow \left.\frac{\partial z}{\partial x}\right|_{(1,0)} = 3$$

$$\frac{\partial z}{\partial y} = -3x + x\,e^{xy} \Rightarrow \left.\frac{\partial z}{\partial y}\right|_{(1,0)} = -2$$

$$dz = \left.\frac{\partial z}{\partial x}\right|_{(1,0)} dx + \left.\frac{\partial z}{\partial y}\right|_{(1,0)} dy$$

$$= 3\,(0.95-1) + (-2)\,(0.12-0)$$

$$= 3\,(-0.05) - 2\,(0.12)$$

$$= \boxed{-0.39}. \cong \text{change in altitude.}$$

2. [**12 points**] Consider the function:

$$f(x, y) = xy^2 - 2x + 2$$

(a) Find and classify each of its critical points as a local minimum, local maximum, or saddle point.

$$\begin{cases} f_x(x,y) = y^2 - 2 = 0 & \Leftrightarrow y = \pm\sqrt{2} \\ f_y(x,y) = 2xy = 0 & \Leftrightarrow x = 0 \text{ or } \cancel{y=0} \end{cases}$$

$\leftarrow$ can not have this because $f_x \neq 0$ when $y=0$

We have 2 critical points: $\boxed{(0, \pm\sqrt{2})}$

$$\left.\begin{array}{l} f_{xx}(x,y) = 0 \\ f_{xy}(x,y) = f_{yx}(x,y) = 2y \\ f_{yy}(x,y) = 2x \end{array}\right\} \Rightarrow D(x,y) = \begin{vmatrix} 0 & 2y \\ 2y & 2x \end{vmatrix} = -4y^2$$

Since $D(0, \pm\sqrt{2}) = -4(\pm\sqrt{2})^2 = -8 < 0$

$\boxed{\text{both critical points are SADDLE POINTS}}$

(b) Find the abolute maximimum value of this function on the region $D = \{(x, y) | x^2 + y^2 \leq 1\}$.

1) no critical points inside the region

2) Boundary: $x^2 + y^2 = 1 \Rightarrow \boxed{y^2 = 1 - x^2}$

$g(x) := f(x, \pm\sqrt{1-x^2}) = x(1-x^2) - 2x - 2 = -x^3 - x + 2$

$g'(x) = -3x^2 - 1$ $\leftarrow$ never zero $\Rightarrow$ no CP's on the boundary.

3) Endpoints: The domain of $y^2 = 1 - x^2$ is $-1 \leq x \leq 1$

So the endpoints are $x = -1 \Rightarrow y = 0 \Rightarrow (-1, 0)$

$x = 1 \Rightarrow y = 0 \Rightarrow (1, 0)$.

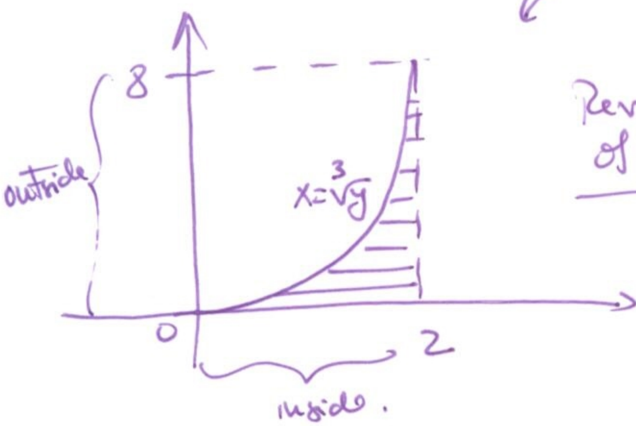
$f(-1, 0) = 4$

$f(1, 0) = 0$

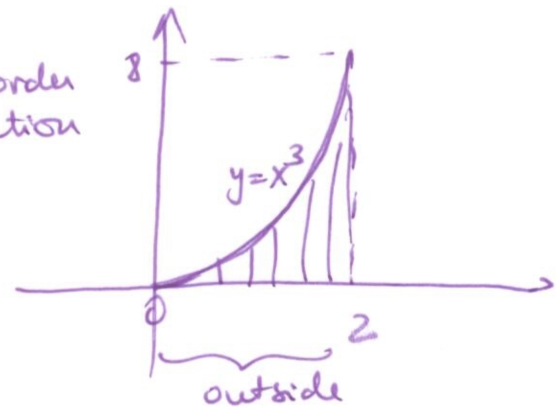
$\boxed{\text{Hence max. value is } z = 4}$. at $(x,y) = (-1, 0)$

3. [**8 points**] Evaluate:

$$\int_0^8 \int_{y^{\frac{1}{3}}}^2 \frac{y^2 e^{x^2}}{x^8} dx dy$$



Reverse order of integration



$$\int_0^8 \int_{\sqrt[3]{y}}^2 \frac{y^2 e^{x^2}}{x^8} dx\, dy = \int_0^2 \int_0^{x^3} \frac{y^2 e^{x^2}}{x^8} dy\, dx$$

$$= \int_0^2 \frac{e^{x^2}}{x^8} \left(\frac{y^3}{3}\right)\Big|_0^{x^3} dx$$

$$= \frac{1}{3}\int_0^2 \frac{e^{x^2}}{x^8} (x^9 - 0)\, dx$$

$$= \frac{1}{3} \int_0^2 x e^{x^2} dx \qquad \boxed{\begin{array}{c} u = x^2 \\ \frac{1}{2}du = x dx \end{array}}$$

$$= \frac{1}{3}\int_0^4 \frac{1}{2} e^u\, du$$

$$= \frac{1}{6} e^u \Big|_0^4 = \boxed{\frac{1}{6}(e^4 - 1)}$$