1. (12 pts) A constant force $\mathbf{F} = \langle 12, -9, 0\rangle$ acts on a particle of mass $m = 3$. Newton's Law says that $\mathbf{F} = m\mathbf{a}$, where $\mathbf{a}$ is the acceleration. At the time $t = 0$, the particle is located at $(0, 0, 7)$ and the initial velocity is $\langle 1, 3, -2\rangle$.

(a) Find the location of the particle at time $t = 1$.

$\vec{a}(t) = \frac{1}{m}\vec{F} = \langle 4, -3, 0\rangle$

$\vec{v}(t) = \int \vec{a}(t)\,dt = \langle 4t + c_1, -3t + c_2, c_3\rangle$

$\vec{v}(0) = \langle 1, 3, -2\rangle \Rightarrow \vec{v}(t) = \langle 4t+1, -3t+3, -2\rangle$

$\vec{r}(t) = \int \vec{v}(t)\,dt = \langle 2t^2 + t + d_1, -\frac{3}{2}t^2 + 3t + d_2, -2t + d_3\rangle$

$\vec{r}(0) = \langle 0, 0, 7\rangle \Rightarrow \vec{r}(t) = \langle 2t^2 + t, -\frac{3}{2}t^2 + 3t, -2t + 7\rangle$

$\boxed{\vec{r}(1) = \langle 3, 3/2, 5\rangle}$

(b) Find the curvature at time $t = 0$.

$$K(0) = \frac{|\vec{r}\,'(0) \times \vec{r}\,''(0)|}{|\vec{r}\,'(0)|^3}$$

$$\langle 1, 3, -2\rangle \times \langle 4, -3, 0\rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & -2 \\ 4 & -3 & 0 \end{vmatrix} = (0-6)\vec{i} - (0 - -8)\vec{j} + (-3-12)\vec{k}$$
$$= \langle -6, -8, -15\rangle$$

$$K(0) = \frac{\sqrt{(-6)^2 + (-8)^2 + (-15)^2}}{(1^2 + 3^2 + (-2)^2)^{3/2}} = \boxed{\frac{\sqrt{325}}{14^{3/2}}} \approx 0.3441514684$$

(c) Find all times when the tangential component of acceleration is zero.

$$a_T = \frac{\vec{r}\,'(t) \cdot \vec{r}\,''(t)}{|\vec{r}\,'(t)|} \stackrel{?}{=} 0 \iff \vec{r}\,'(t) \cdot \vec{r}\,''(t) \stackrel{?}{=} 0$$

$$4(4t+1) - 3(-3t+3) + 0 \stackrel{?}{=} 0$$

$$16t + 4 + 9t - 9 = 0$$

$$25t = 5 \Rightarrow t = 5/25 = \boxed{1/5}$$

2. (12 pts)

(a) Using the linear approximation at $(2, 1)$, estimate the value of the function

$$f(x, y) = y\cos(2 - xy) + x^2 + \ln(y)$$

at the point $(2.1, 0.95)$.

$f(2,1) = 1\cos(0) + 4 + 0 = 5$

$f_x = -(-y)y\sin(2-xy) + 2x \Rightarrow f_x(2,1) = 1^2\sin(0) + 2(2) = 4$

$f_y = \cos(2-xy) - (-x)y\sin(2-xy) + \frac{1}{y} \Rightarrow f_y(2,1) = \cos(0) + 2\sin(0) + \frac{1}{1} = 2$

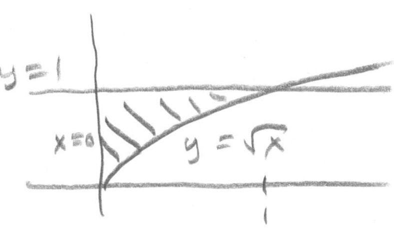
$L(x,y) = 5 + 4(x-2) + 2(y-1)$

$L(2.1, 0.95) = 5 + 4(0.1) + 2(-0.05) = 5 + 0.4 - 0.1 = \boxed{5.3}$

ACTUAL VALUE
$\approx 5.308694831$

(b) Find the volume of the solid that is below the surface $z + x^2 = 4$, above $z = 0$, and between the surfaces $x = 0$, $y = \sqrt{x}$, and $y = 1$.

INTEGRAND: $z = 4 - x^2$




$\iint_D 4 - x^2 \, dA$

$= \int_0^1 \int_{\sqrt{x}}^1 4 - x^2 \, dy\, dx$ or $\int_0^1 \int_0^{y^2} 4 - x^2 \, dx\, dy$

$= \int_0^1 4x - \frac{1}{3}x^3 \Big|_0^{y^2} dy$

$= \int_0^1 4y^2 - \frac{1}{3}y^6 \, dy$

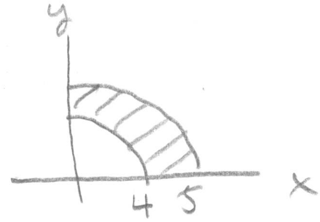
$= \frac{4}{3}y^3 - \frac{1}{21}y^7 \Big|_0^1$

$= \frac{4}{3} - \frac{1}{21} = \frac{28-1}{21} = \boxed{\frac{27}{21} = \frac{9}{7}}$

$= 1.\overline{285714}$

3. (12 pts)

(a) Consider the region $D$ bounded between $x^2+y^2=16$ and $x^2+y^2=25$ in the **first quadrant**. Evaluate $\displaystyle\iint_D \frac{y^2\cos(\sqrt{x^2+y^2})}{(x^2+y^2)^{3/2}}\,dA$




$$\int_0^{\pi/2}\int_4^5 \frac{r^2\sin^2\theta\cos(r)}{r^3}\, r\,dr\,d\theta$$

$$\int_0^{\pi/2}\sin^2\theta\,d\theta \int_4^5 \cos(r)\,dr$$

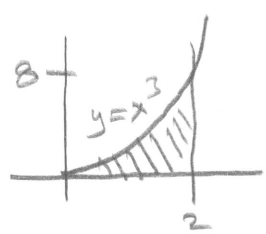
$$\int_0^{\pi/2}\frac{1}{2}(1-\cos(2\theta))\,d\theta \quad \int_4^5 \cos(r)\,dr$$

$$\left(\frac{1}{2}\theta-\frac{1}{4}\sin(2\theta)\Big|_0^{\pi/2}\right)\cdot\left[\sin(r)\Big|_4^5\right)$$

$$\left[\left(\frac{\pi}{4}-0\right)-(0-0)\right]\cdot\left[\sin(5)-\sin(4)\right] = \boxed{\frac{\pi}{4}(\sin(5)-\sin(4))}$$

$\approx -0.1587460743$

(b) By switching the order of integration, evaluate $\displaystyle\int_0^8\int_{\sqrt[3]{y}}^2 \sqrt{x^4+1}\,dx\,dy$.




$$\int_0^2\int_0^{x^3}\sqrt{x^4+1}\,dy\,dx$$

$$\int_0^2 y\sqrt{x^4+1}\Big|_0^{x^3}dx$$

$$\int_0^2 x^3\sqrt{x^4+1}\,dx$$

$u=x^4+1$

$du=4x^3\,dx$

$dx=\frac{1}{4x^3}du$

$$\int_1^{17}\frac{1}{4}u^{1/2}\,du = \frac{1}{4}\cdot\frac{2}{3}u^{3/2}\Big|_1^{17} = \boxed{\frac{1}{6}(17^{3/2}-1)}$$

$\approx 11.51546594$

4. (12 pts) Find non-negative numbers $x$, $y$ and $z$ that minimize the quantity

$$A = x^2 + 3y^2 + z,$$

subject to the condition $xyz = 12$.
Give $x$, $y$, and $z$ as decimals correct to 3 digits after the decimal.
(Use the second derivative test and your decimal answers at the end of the problem to verify your answer is a local minimum).

$$A(x,y) = x^2 + 3y^2 + \frac{12}{xy}$$

① $A_x = 2x - \frac{12}{x^2 y} \overset{?}{=} 0 \Rightarrow 2x^3 y = 12 \Rightarrow x^3 y = 6 \Rightarrow y = \frac{6}{x^3}$

② $A_y = 6y - \frac{12}{xy^2} \overset{?}{=} 0 \Rightarrow 6xy^3 = 12 \Rightarrow xy^3 = 2$

① & ② $\Rightarrow x\left(\frac{6}{x^3}\right)^3 = 2 \Rightarrow \frac{6^3}{x^8} = 2$

$\Rightarrow x^8 = 108$

$$\boxed{\begin{aligned} x &= 108^{1/8} \approx 1.795 \\ y &= \frac{6}{108^{3/8}} \approx 1.037 \\ z &= \frac{12}{xy} = \frac{12}{108^{1/8}\left(\frac{6}{108^{3/8}}\right)} = 2(108)^{1/4} \\ &\approx 6.447 \end{aligned}}$$

$A_{xx} = 2 + \frac{24}{x^3 y}$

$A_{yy} = 6 + \frac{24}{xy^3}$

$A_{xy} = \frac{12}{x^2y^2}$

at the critical value

$A_{xx} = 6$

$A_{yy} = 18$

$A_{xy} \approx 3.46410161$

$D = A_{xx}A_{yy} - (A_{xy})^2 = 6 \cdot 18 - (3.46410161)^2 = 96 > 0$

and $A_{xx} = 6 > 0$

LOCAL MINIMUM