Instructions:

- Your exam consists of FIVE problems. Please check that you have all four of them.

- No books or notebooks allowed; you may use an A4 double-sided, handwritten sheet of notes for personal use (do not share).

- Place a box around your final answer to each question.

- No graphing calculators allowed (scientific calculators OK).

- Answers with little or no justification may receive no credit.

- Answers obtained by guess-and-check work will receive little or no credit, even if correct.

- Read problems carefully.

- Raise your hand if you have a question.

- If you need more space, use additional blank sheets which will be provided by your TA. It is your responsibility to have him/her staple the additional sheets to your exam before you turn it in.

- Please turn off cell phones. GOOD LUCK!
Problem 1. (10 pts.) Find the double integral of the function \( f(x, y) = xy \) over the triangle with vertices \((0, 0), (1, 0), \) and \((1, 1)\).

Solution. We get
\[
\iint f dA = \int_0^1 \int_0^x xy \, dy \, dx = \frac{1}{2} \int_0^1 x^3 \, dx = \frac{1}{8}.
\]
Problem 2. (10 pts) Find equation of the tangent plane of the function \( f(x, y) = \ln(x - 3y) \) at \((7, 2)\). Use it to approximate \( f(6.9, 2.02) \).

Solution. Let \( x_0 = 7 \), \( y_0 = 2 \), and \( z_0 = 0 \). Also
\[
f_x(7, 2) = \frac{1}{x_0 - 3y_0} = 1, \quad f_y(7, 2) = \frac{-3}{x_0 - 3y_0} = -3.
\]
Thus the tangent plane is given by
\[
z = (x - 7) - 3(y - 2) = x - 3y - 1.
\]
Moreover
\[
f(6.9, 2.02) \approx 0 + 1(6.9 - 7) - 3(2.02 - 2) = -.1 - .06 = -.16.
\]
Problem 3. (10 pts) Answer the following questions.

(i) At what point on the curve \( x = t^3, y = 3t, z = t^4 \) is the normal plane parallel to the plane \( 6x + 6y - 8z = 1 \)?

Solution. Let \( \mathbf{r}(t) = (x(t), y(t), z(t)) \). We want to find \( t \) such that \( \mathbf{r}'(t) \) is a parallel to the normal vector of the given plane, i.e., \( (6, 6, -8) \). Now \( \mathbf{r}'(t) = (3t^2, 3, 4t^3) \). Thus, \( t = -1 \) is the only solution. Hence the point is \((-1, -3, 1)\).

(ii) Suppose the location of a particle at time \( t \) is given by the curve \( \mathbf{r}(t) = (t, t^2, t) \). At what point on its trajectory is the normal component of its acceleration maximum?

We know
\[
a_N = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|}.
\]

We see
\[
\mathbf{r}'(t) = (1, 2t, 1), \quad \mathbf{r}''(t) = (0, 2, 0), \quad \mathbf{r}'(t) \times \mathbf{r}''(t) = (-2, 0, 2).
\]
Thus
\[
a_N = \frac{\sqrt{8}}{\sqrt{2 + 4t^2}}.
\]
This is maximum at \( t = 0 \), i.e., at \((0, 0, 0)\).
Problem 4. (10 pts) Find the absolute maximum and minimum values of the function $f(x, y) = x + y$ in the domain $D = \{(x, y) : 1 \leq x^2 + y^2 \leq 4\}$.

Solution. First find critical points inside the domain by solving
\[
\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0.
\]
There is no solution. So we move to the boundary. There is an outer and an inner circular boundary.

The inner boundary can be parametrized by $(\cos t, \sin t), 0 \leq t < 2\pi$. The function is now
\[
f(t) = \cos(t) + \sin(t), \quad f'(t) = -\sin(t) + \cos(t) = 0 \Rightarrow t = \frac{\pi}{4}, \text{ or, } \frac{5\pi}{4}.
\]
We evaluate $f(\pi/4) = \sqrt{2}$, $f(5\pi/4) = -\sqrt{2}$.

The outer boundary can be parametrized by $(2\cos t, 2\sin t), 0 \leq t < 2\pi$. Here
\[
f(t) = 2\sin(t) + 2\cos(t), \quad f'(t) = 2\sin(t) - 2\cos(t) = 0.
\]
The same solution gives us $f(\pi/4) = 2\sqrt{2}$, $f(5\pi/4) = -2\sqrt{2}$.

Thus the absolute maximum is $2\sqrt{2}$ and the absolute minimum is $-2\sqrt{2}$. 
Problem 5. (10 pts) Use polar coordinates to find the volume of the solid that is inside the sphere $x^2 + y^2 + z^2 = 16$ and outside the cylinder $x^2 + y^2 = 4$.

Solution. The volume between the two bodies is

$$I = 2 \left[ \int \int_{x^2+y^2 \leq 16} \sqrt{16 - x^2 - y^2} dA - \int \int_{x^2+y^2 \leq 4} \sqrt{16 - x^2 - y^2} dA \right]$$

Using polar coordinates

$$\frac{I}{2} = \int_0^{2\pi} \int_0^4 r \sqrt{16 - r^2} dr d\theta - \int_0^{2\pi} \int_0^2 \sqrt{16 - r^2} dr d\theta$$

$$= 2\pi \left[ -\frac{1}{3} (16 - r^2)^{3/2} \right]_0^4 = \frac{2\pi}{3} 12^{3/2}.$$}

Thus $I = 4\pi 12^{3/2} / 3$. 

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