SOLUTIONS

1 (11 points total) All the parts of this problem concern the vector function $\mathbf{r}(t)$ that satisfies the following conditions: the acceleration is $\mathbf{a}(t) = 6t\mathbf{i} + \mathbf{j} - 12t^2\mathbf{k}$ and the initial position and velocity are given by $\mathbf{r}(0) = 2\mathbf{k}$ and $\mathbf{v}(0) = \mathbf{i} + \mathbf{j}$.

(a) (**4 points**) Find the vector function $\mathbf{r}(t)$.

$$\vec{v}\,'(t) = \vec{a}(t) = \langle 6t, 1, -12t^2 \rangle$$

$$\vec{v}(t) = \langle 3t^2 + c_1, t + c_2, -4t^3 + c_3 \rangle = \langle 3t^2+1, t+1, -4t^3 \rangle$$

$$\vec{v}(0) = \langle 1, 1, 0 \rangle \Rightarrow c_1 = 1, c_2 = 1, c_3 = 0$$

$$\vec{r}\,'(t) = \vec{v}(t) = \langle 3t^2+1, t+1, -4t^3 \rangle \Rightarrow \vec{r}(t) = \langle t^3 + t + d_1, \tfrac{t^2}{2} + t + d_2, -t^4 + d_3 \rangle$$

$$\vec{r}(0) = \langle 0, 0, 2 \rangle \Rightarrow \boxed{\vec{r}(t) = \langle t^3 + t, \tfrac{t^2}{2} + t, 2 - t^4 \rangle}$$

(b) (**3 points**) Write an equation of the normal plane to the curve described by $\mathbf{r}(t)$ at the point where $t = 0$.

$$\vec{n} = \vec{r}\,'(0) = \vec{v}(0) = \langle 1, 1, 0 \rangle, \quad \vec{r}(0) = \langle 0, 0, 2 \rangle$$

$$1(x-0) + 1(y-0) + 0(z-2) = 0$$

$$\boxed{x + y = 0}$$

(c) (**4 points**) Compute the curvature of the curve described by $\mathbf{r}(t)$ at $t = 0$.

$$K(0) = \frac{|\vec{r}\,'(0) \times \vec{r}\,''(0)|}{|\vec{r}\,'(0)|^3} = \frac{|\vec{v}(0) \times \vec{a}(0)|}{|\vec{v}(0)|^3}$$

$$\vec{v}(0) \times \vec{a}(0) = (\vec{i} + \vec{j}) \times \vec{j} = \vec{k}, \quad |\vec{v}(0)| = |\langle 1,1,0 \rangle| = \sqrt{2}$$

$|\vec{k}| = 1$, so

$$K = K(0) = \frac{1}{(\sqrt{2})^3} = \boxed{2^{-3/2}}$$

2 (9 points) Find the tangent plane to the surface given by the graph of

$$f(x,y) = \sqrt{22 - x^2 - 2y^2}$$

at $(2, 1)$. Use the linear approximation to estimate $f(1.98, 0.96)$.

$$f_x(x,y) = \left.\frac{-x}{\sqrt{22-x^2-2y^2}}\right|_{(2,1)} = \frac{-2}{\sqrt{22-4-2}} = -\frac{1}{2}$$

$$f_y(x,y) = \left.\frac{-2y}{\sqrt{22-x^2-2y^2}}\right|_{(2,1)} = \frac{-2}{\sqrt{22-4-2}} = -\frac{1}{2}$$

$$f(2,1) = \sqrt{16} = 4$$

tangent plane: $\boxed{z - 4 = -\frac{1}{2}(x-2) - \frac{1}{2}(y-1)}$

or

$$\boxed{x + y + 2z = 11}$$

$$f(1.98, 0.96) \approx 4 - \frac{1}{2}(1.98-2) - \frac{1}{2}(0.96-1)$$

$$= 4 + 0.01 + 0.02 = \boxed{4.03}$$

3 (10 points) Find three positive numbers x, y, and z whose sum is 12 and for which the product

$$xyz^2$$

is a maximum.

maximize xyz^2 subject to $x+y+z=12$

Eliminate x: $x = 12 - y - z$

$f(y,z) = (12-y-z)yz^2$; want to maximize it over $y>0, z>0, 12-y-z>0$.

$f_y(y,z) = -yz^2 + (12-y-z)z^2 = 0$ (1)

$f_z(y,z) = -yz^2 + (12-y-z)2yz = 0$ (2)

Since $y, z \neq 0$, we obtain $(1) \Rightarrow -y + 12 - y - z = 0$

$(2) \Rightarrow -z + (12-y-z)2 = 0$

$$\left.\begin{array}{l} 2y+z=12 \\ 2y+3z=24 \end{array}\right| \Rightarrow z=6 \Rightarrow y=3 \Rightarrow x=3$$

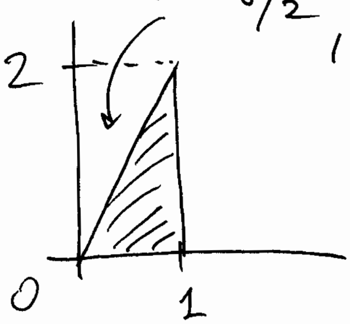
Since this is the only critical point, it must be the maximum: $\boxed{x=3, y=3, z=6}$

4 (10 points total)

(a) (4 points) Change the order of integration in the following integral:

$$\int_0^2 \int_{y/2}^1 x^2 \sin(xy)\, dx\, dy = \int_0^1 \int_0^{2x} x^2 \sin(xy)\, dy\, dx$$

$x = y/2$, $y = 2x$



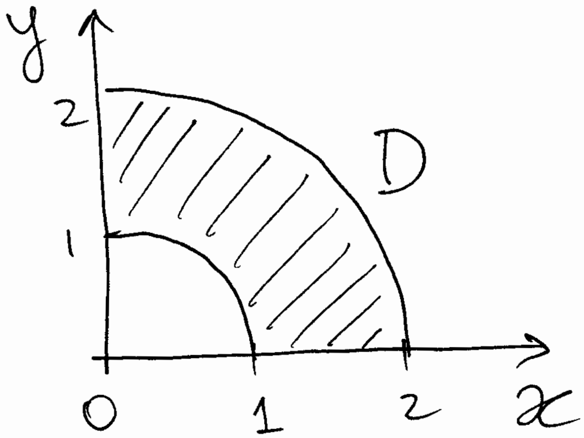
(b) (6 points) Evaluate the integral.

$$\int_0^2 \int_{y/2}^1 x^2 \sin(xy)\, dx\, dy = \int_0^1 \int_0^{2x} x^2 \sin(xy)\, dy\, dx$$

$$= \int_0^1 x^2 \left[-\frac{\cos(xy)}{x}\right]\Bigg|_{y=0}^{2x} dx = \int_0^1 x\left[1 - \cos(2x^2)\right] dx$$

$$= \left(\frac{x^2}{2} - \frac{\sin(2x^2)}{4}\right)\Bigg|_0^1 = \boxed{\frac{1}{2} - \frac{\sin(2)}{4}}$$

5 (10 points) Find the volume of the solid between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ in the first octant, bounded above by $z = x + y$ and below by $z = 0$.



$$V = \iint_D (x+y)\, dA$$

in polar coordinates:

$$V = \int_0^{\pi/2}\int_1^2 (r\cos\theta + r\sin\theta)\, r\, dr\, d\theta$$

$$= \int_0^{\pi/2} (\cos\theta + \sin\theta) \frac{r^3}{3}\Big|_1^2 = \frac{8-1}{3}(\sin\theta - \cos\theta)\Big|_0^{\pi/2}$$

$$= \frac{7}{3}(1-(-1)) = \boxed{\frac{14}{3}}$$