Math 126 A - Winter 2019
Midterm Exam Number One
February 7, 2019

Name: ______________________ Student ID no. : ______________

Signature: ____

1	10	
2	10	
3	10	
4	10	
5	10	
Total	50	

- This exam consists of **five** problems on **four** double-sided pages.
- Show all work for full credit.
- You may use a TI-30X IIS calculator during this exam. Other calculators and electronic devices are not permitted.
- You do not need to simplify your answers.
- If you use a trial-and-error or guess-and-check method when a more rigorous method is available, you will not receive full credit.
- $\boxed{\text{Draw a box}}$ around your final answer to each problem.
- **Do not write within 1 centimeter of the edge!** Your exam will be scanned for grading.
- If you run out of room, write on the back of the first or last page and indicate that you have done so. If you still need more room, raise your hand and ask for an extra page.
- You may use one hand-written double-sided 8.5″ by 11″ page of notes.
- You have 50 minutes to complete the exam.

1. **[5 points per part]** Consider the points $A(2,0,0)$, $B(0,1,0)$ and $C(0,0,3)$.

   (a) Find an equation for the plane passing through the points A, B, C.

   First we find two vectors that lie on the plane, for example

$$\overrightarrow{AB} = \langle -2,1,0\rangle, \overrightarrow{AC} = \langle -2,0,3\rangle.$$

   The cross product of these vectors will yield a vector that is perpendicular to both, namely

$$\overrightarrow{AB} \times \overrightarrow{AC} = 3i + 6j + 2k.$$

   This is a normal vector to our desired plane.
   Since we know that $A(2,0,0)$ is on the plane, the equation of the plane is

$$3(x-2) + 6y + 2z = 0,$$

   or

$$3x + 6y + 2z = 6.$$

   (b) Find the area of the triangle ABC.

   The magnitude of the cross product from part (a) is the area of the parallelogram formed by the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$, and the area of the triangle ABC is half of the area of this parallelogram, so the area of the triangle ABC is given by

$$\frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{\sqrt{3^2+6^2+2^2}}{2} = \frac{\sqrt{49}}{2} = \frac{7}{2}.$$

2. **[5 points per part]** Consider the plane H with equation $3x + 6y + 2z = 6$ and the point $P(0,0,0)$.

   (a) Find the equation of the line passing through P and perpendicular to the plane H.

   Let L be the line described. Since L is perpendicular to H, we can set the direction vector of L equal to the normal vector of the plane, that is, $\langle 3, 6, 2\rangle$. Since the point $(0,0,0)$ is on the line L, the parametric equation of L is given by

$$\langle 3t, 6t, 2t\rangle.$$

   (b) Find the distance from P to H.

   First, we need to find the point of intersection with H. In this case, we have to find the parameter t on the line so that the point $(3t, 6t, 2t)$ lies on the plane H, that is,

$$3(3t) + 6(6t) + 2(2t) = 6.$$

   Solving this equation for t, we get $t = \frac{6}{49}$. So, the point of intersection will be

$$(6/49, 36/49, 12/49).$$

   The distance is

$$\sqrt{(6/49)^2 + (36/49)^2 + (12/49)^2} = \frac{6}{7}.$$

   Alternately, we can use the formula of the distance from a point to a plane to obtain

$$d = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|3 * 0 + 6 * 0 + 2 * 0 - 6|}{\sqrt{9 + 36 + 4}} = \frac{6}{7}.$$

3. **[10 points total]** Consider the point $Q(0,0,0)$ and the plane S with equation $z=2$.

(a) **[8 points]** Find the equation for the set of points equidistant from Q and S.

We want to find the set of points (x,y,z) so that the distance from (x,y,z) to $Q(0,0,0)$ is the same as the distance from $z=2$ to (x,y,z). The distance to $(0,0,0)$ is

$$\sqrt{(x-0)^2+(y-0)^2+(z-0)^2}$$

and the distance to the plane $z=2$ is

$$\sqrt{(x-x)^2+(y-y)^2+(z-2)^2}.$$

Setting these equal to each other, and squaring, we get

$$x^2+y^2+z^2=(z-2)^2.$$

Reorganizing this, we get

$$x^2+y^2=-4z+4$$

or

$$\frac{x^2}{4}+\frac{y^2}{4}-1=\frac{z}{-1}.$$

(b) **[2 points]** What kind of surface is this?

This is the equation of an elliptic (in fact, circular) paraboloid.

4. **[5 points each part]**

(a) Find a vector function that describes the intersection of the surfaces $x = \cos z$ and $x^2 + y^2 = 1$.

Setting $z = t$, we obtain $x = \cos t$ from the equation of the first surface. Using the equation of the second surface,

$$x^2 + y^2 = 1,$$

we have $y = \sin t$, since

$$\cos^2 t + \sin^2 t = 1.$$

So the vector function is

$$\overrightarrow{\mathbf{r}}(t) = \langle \cos t, \sin t, t \rangle,$$

a helix.

(b) Compute the curvature of this curve. Hint: it does not matter which point on the curve you pick!

We use the curvature formula

$$\kappa(t) = \frac{|\overrightarrow{\mathbf{r}}'(t) \times \overrightarrow{\mathbf{r}}''(t)|}{|\overrightarrow{\mathbf{r}}'(t)|^3}.$$

Using our answer above, we have

$$\overrightarrow{\mathbf{r}}'(t) = \langle -\sin(t), \cos(t), 1 \rangle$$

and

$$\overrightarrow{\mathbf{r}}''(t) = \langle -\cos(t), -\sin(t), 0 \rangle.$$

Taking the cross product we get

$$\overrightarrow{\mathbf{r}}'(t) \times \overrightarrow{\mathbf{r}}''(t) = \langle \sin(t), -\cos(t), 1 \rangle.$$

So

$$|\overrightarrow{\mathbf{r}}'(t) \times \overrightarrow{\mathbf{r}}''(t)| = \sqrt{(\sin(t))^2 + (-\cos(t))^2 + 1} = \sqrt{2}.$$

We have

$$|\overrightarrow{\mathbf{r}}'(t)|^3 = (\sqrt{(-\sin(t))^2 + (\cos(t))^2 + 1^2})^3 = (\sqrt{2})^3.$$

so

$$\kappa(t) = \frac{|\overrightarrow{\mathbf{r}}'(t) \times \overrightarrow{\mathbf{r}}''(t)|}{|\overrightarrow{\mathbf{r}}'(t)|^3} = \frac{\sqrt{2}}{(\sqrt{2})^3} = \frac{1}{2}.$$

5. **[5 points each part]**

(a) Find the point of intersection between the vector functions $\overrightarrow{\mathbf{r}}(t) = \langle \cos t, \sin t, t \rangle$ and the line L with vector equation $\langle s, 1-s, \pi/2 \rangle$.

At the point of intersection, $\cos(t) = s$, $\sin(t) = 1-2$, and $t = \frac{\pi}{2}$ (the x, y, z coordinates are equal). This gives $t = \frac{\pi}{2}$, $s = \cos\frac{\pi}{2} = 0$, and $1 - s = 1 = \sin(\pi/2)$.
Thus, the intersection occurs at the point $(0, 1, \pi/2)$.

(b) Find the angle between the curve $\overrightarrow{\mathbf{r}}(t)$ and the line L at the point of intersection from part (a).

The tangent vector to $\overrightarrow{\mathbf{r}}(t)$ is given by

$$\overrightarrow{\mathbf{r}}'(t) = \langle -\sin(t), \cos(t), 1 \rangle.$$

At $t = \pi/2$, this gives

$$\overrightarrow{\mathbf{r}}'(\pi/2) = \langle -1, 0, 1 \rangle.$$

The tangent vector to the line at any point is given by

$$\langle 1, -1, 0 \rangle$$

. To compute the angle θ between these two vectors, we use the dot product, and the formula

$$\overrightarrow{\mathbf{r}}'(\pi/2) \cdot \langle 1, -1, 0 \rangle = |\overrightarrow{\mathbf{r}}'(\pi/2)| \cdot |\langle 1, -1, 0 \rangle| \cos\theta$$

That is,

$$\langle -1, 0, 1 \rangle \cdot \langle 1, -1, 0 \rangle = |\langle -1, 0, 1 \rangle| \cdot |\langle 1, -1, 0 \rangle| \cos\theta$$

We have

$$|\langle -1, 0, 1 \rangle| = \sqrt{(-1)^2 + 0^2 + 1^2} = \sqrt{2},$$
$$|\langle 1, -1, 0 \rangle| = \sqrt{1^2 + (-1)^2 + 0^2} = \sqrt{2},$$

and

$$\langle -1, 0, 1 \rangle \cdot \langle 1, -1, 0 \rangle = (-1)1 + 0(-1) + 1(0) = -1.$$

So

$$\cos\theta = -\frac{1}{2},$$

so

$$\theta = \arccos\left(-\frac{1}{2}\right).$$