Math 126, Section C - Winter 2015
Midterm I
February 3, 2015

Name: ____________________________________
Student ID Number: ______________
Section: CA 11:30-12:20 by Sam ☐ CB 12:30-1:20 by Sam ☐
CC 11:30-12:20 by Ru-Yu ☐ CD 12:30-1:20 by Ru-Yu ☐

exercise	**possible**	**score**
1	7	
2	11	
3	8	
4	12	
5	12	
total	50	

- Check that this booklet has all the exercises indicated above.
- TURN OFF YOUR CELL PHONE.
- Write your name and your student ID.
- This is a 50 minute test.
- You may use a scientific calculator and one 8.5×11 inch sheet of (two-sided) handwritten notes. All other electronic devices (including graphing calculators) are forbidden.
- Unless otherwise indicated, your answers should be exact instead of decimal approximations. For example $\frac{\pi}{4}$ is an exact answer and is preferable to its decimal approximation 0.78.
- Unless otherwise indicated, show your work and justify all your answers. Box your final answer.

Sample solution

Exercise 1 (7 points).

Consider the points $A = (3,4,1)$, $B = (4,-1,0)$ and $C = (1,2,2)$. What is the area of the triangle ABC?

Solution: We have $\overrightarrow{AB} = (1,-5,-1)$ and $\overrightarrow{AC} = (-2,-2,1)$. Then

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -5 & -1 \\ -2 & -2 & 1 \end{vmatrix} = (-5-2, -(1-2), -2-10) = (-7,1,-12)$$

The area is then

$$\frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2}\sqrt{7^2+1^2+12^2} = \boxed{\frac{1}{2}\sqrt{194} \approx 6.9642}$$

Exercise 2 (11 points).

Consider the two lines given by symmetric equations

$$\ell_1: \ x-1=y+2=z-3 \qquad \ell_2: \ \frac{x-4}{2}=y=z-5$$

(a) Both lines intersect in exactly one point. Compute the angle of the intersection (rounded to the nearest degree).

Solution:

- Line ℓ_1 contains for example the points $(1,-2,3)$ and $(2,-1,4)$. Hence the direction vector the line is $(1,1,1)$.
- Line ℓ_2 contains for example the points $(4,0,5)$ and $(2,-1,4)$ and hence the direction vector is $(2,1,1)$. The angle θ between both direction vectors satisfies

  $$\cos(\theta)=\frac{(1,1,1)\cdot(2,1,1)}{|(1,1,1)|\cdot|(2,1,1)|}=\frac{4}{\sqrt{3}\cdot\sqrt{6}}=\frac{4}{\sqrt{18}}$$

  Hence $\theta=\cos^{-1}(\frac{4}{\sqrt{18}})\approx 19.4712^o$ which can be rounded to $\boxed{19^o}$ (0.3398 in radiants).

(In fact, $(2,-1,4)$ is the unique point where ℓ_1 and ℓ_2 intersect, but we didn't ask for that one.)

(b) Find the equation of the plane that contains both lines ℓ_1 and ℓ_2.
Solution: We already learned that the plane contains the directions $(1,1,1)$ and $(2,1,1)$. Their cross product is

$$(1,1,1)\times(2,1,1)=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{vmatrix}=(0,-(1-2),1-2)=(0,1,-1)$$

Since $(4,0,5)\cdot(0,1,-1)=-5$, the equation of the plane is

$$\boxed{y-z=-5}$$

Exercise 3 (2+2+2+2=8 points).

We want to study the surface in $\mathbb{R}^3$ that is described by the equation

$$\frac{x^2}{9} - \frac{y^2}{4} + \frac{z^2}{16} + 5 = 0$$

a) Fill out the following table (no justification needed).

	a parabola	a hyperbola	an ellipse	empty	other
The trace with the xy-plane is	☐	X	☐	☐	☐
The trace with the plane $y = 2$ is	☐	☐	☐	X	☐
The trace with the plane $y = 6$ is	☐	☐	X	☐	☐

$b)$ Hence, the surface is an

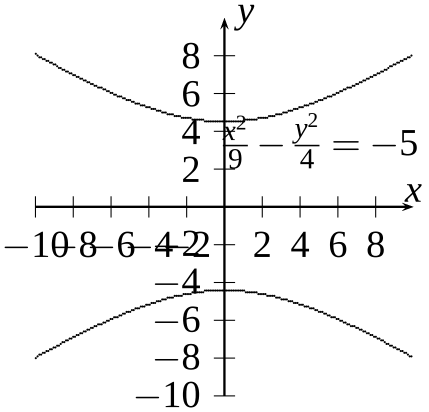
- ☐ elliptic cylinder
- ☐ parabolic cylinder
- ☐ hyperbolic cylinder
- ☐ ellipsoid
- ☐ cone
- ☐ elliptic paraboloid
- ☐ hyperboloid of one sheet
- X hyperboloid of two sheets

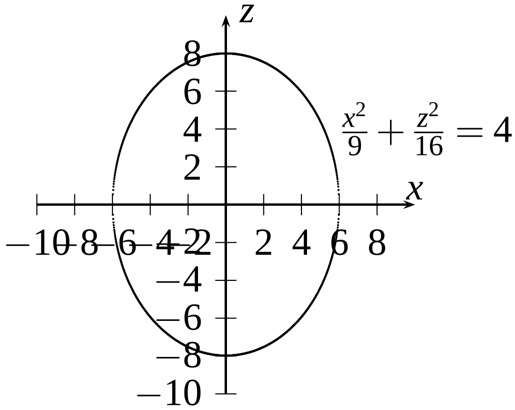
Solution:

- The trace with the xy-plane is $\frac{x^2}{9} - \frac{y^2}{4} = -5$ which is a **hyperbola**.
- The trace with the plane $y = 2$ is

$$\frac{x^2}{9} + \frac{z^2}{16} = -4$$

  which has **no solution**.
- The trace with the plane $y = \pm 6$ is $\frac{x^2}{9} + \frac{z^2}{16} = 4$ which is an **ellipse**.
- Overall the surface is a **hyperboloid of two sheets**.





Exercise 4 (4+8=12 points).

The equation $r = 2\theta + 1$ for $\theta \geq 0$ describes a curve in $\mathbb{R}^2$ in polar coordinates.

$a)$ List 3 points (in Cartesian coordinates) where the curve intersects the positive y-axis.

Solution: The first 3 points are

angle	point
$\theta = \pi/2$	$(0, \pi+1) \approx (0, 4.1416)$
$\theta = (5/2)\pi$	$(0, 5\pi+1) \approx (0, 16.70796)$
$\theta = (9/2)\pi$	$(0, 9\pi+1) \approx (0, 29.2743)$

$b)$ Consider the line that is tangent to the curve at $\theta = \pi$. What is its slope?

Solution: In Cartesian coordinates, the curve can be described as the vector function

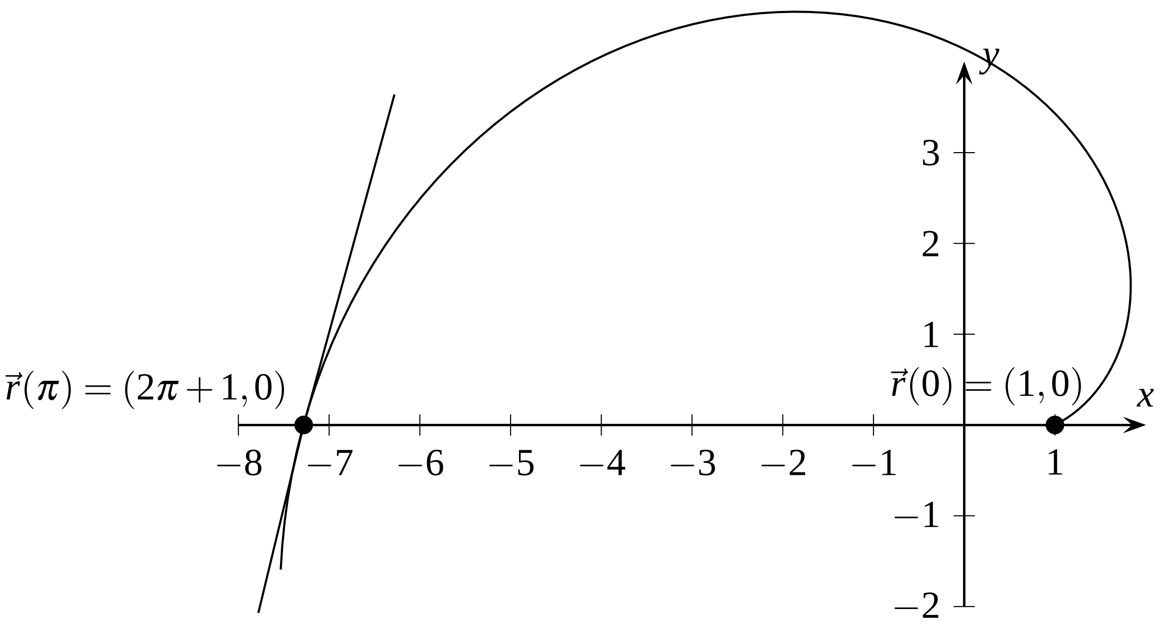
$$\vec{r}(\theta) = (x(\theta), y(\theta)) = ((2\theta+1)\cdot\cos(\theta), (2\theta+1)\cdot\sin(\theta)).$$

The derivative vector is

$$\vec{r}\,'(\theta) = (x'(\theta), y'(\theta)) = (2\cos(\theta) - (2\theta+1)\sin(\theta), 2\sin(\theta) + (2\theta+1)\cos(\theta)).$$

and $\vec{r}\,'(\pi) = (-2, -2\pi - 1)$. The slope at $\theta = \pi$ is

$$\frac{y'(\pi)}{x'(\pi)} = \frac{-2\pi - 1}{-2} = \pi + \frac{1}{2}$$



Exercise 5 (12 points).

Compute the curvature $\kappa(t)$ for the curve $\vec{r}(t) = (t + \sin(t), \frac{t^3}{\pi}, \cos(3t))$ at $t = \frac{\pi}{2}$ (I prefer an exact answer).

Solution: We compute

$$\begin{aligned}
\vec{r}\,'(t) &= (1 + \cos(t), \frac{3t^2}{\pi}, -3\sin(3t)) \\
\vec{r}\,'(\frac{\pi}{2}) &= (1, \frac{3}{4}\pi, 3) \\
|\vec{r}\,'(\frac{\pi}{2})| &= \sqrt{10 + \frac{9}{16}\pi^2} \\
\vec{r}\,''(t) &= (-\sin(t), \frac{6t}{\pi}, -9\cos(3t)) \\
\vec{r}\,''(\frac{\pi}{2}) &= (-1, 3, 0) \\
\vec{r}\,'(t) \times \vec{r}\,''(t) &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & \frac{3}{4}\pi & 3 \\ -1 & 3 & 0 \end{vmatrix} = (-9, -3, 3 + \frac{3}{4}\pi) \\
|\vec{r}\,'(t) \times \vec{r}\,''(t)| &= \sqrt{99 + \frac{9}{2}\pi + \frac{9}{16}\pi^2}
\end{aligned}$$

Using the formula from the lecture we get

$$\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3} \overset{\text{for } t=\frac{\pi}{2}}{=} \frac{\sqrt{99 + \frac{9}{2}\pi + \frac{9}{16}\pi^2}}{(10 + \frac{9}{16}\pi^2)^{3/2}}$$