Math 126 A - Spring 2019
Midterm Exam Number One Solutions
April 25, 2019

Name: ________________________ Student ID no. : ______________

Signature: ____

1	10	
2	10	
3	10	
4	10	
5	10	
Total	50	

- This exam consists of **five** problems on **four** double-sided pages.
- Show all work for full credit.
- You may use a TI-30X IIS calculator during this exam. Other calculators and electronic devices are not permitted.
- You do not need to simplify your answers.
- If you use a trial-and-error or guess-and-check method when a more rigorous method is available, you will not receive full credit.
- $\boxed{\text{Draw a box}}$ around your final answer to each problem.
- **Do not write within 1 centimeter of the edge!** Your exam will be scanned for grading.
- If you run out of room, write on the back of the first or last page and indicate that you have done so. If you still need more room, raise your hand and ask for an extra page.
- You may use one hand-written double-sided 8.5″ by 11″ page of notes.
- You have 50 minutes to complete the exam.

1. **[10 points]** Find the line of intersection between the two planes $x + y + z = 3$ and $x + 2y + 2z = 5$.

   For any two nontrivially-intersecting planes, their line of intersection must lie in both planes and therefore must have a direction vector orthogonal to each of the planes' normal vectors. Since we are looking for a vector perpendicular to two other vectors, we use the cross product of these two vectors. Next, we need to find a single point on the line of intersection and then we can parametrize the line.

   The normal vectors for the planes can be read off from the equations:

$$\begin{aligned} x + y + z = 3 &\implies \overrightarrow{\mathbf{n}}_1 = \langle 1, 1, 1 \rangle \\ x + 2y + 2z = 5 &\implies \overrightarrow{\mathbf{n}}_2 = \langle 1, 2, 2 \rangle \end{aligned}$$

   This gives the direction vector of the line

$$\overrightarrow{\mathbf{v}} := \overrightarrow{\mathbf{n}}_1 \times \overrightarrow{\mathbf{n}}_2 = \langle 0, -1, 1 \rangle.$$

   Now we need to find a point that lies on both planes. By inspection, the point $P = (1, 1, 1)$ satisfies the equations of both planes. Thus we have a point and a direction vector, which is the information we need to parametrize the line of intersection:

$$\ell(t) = \langle 1, 1 - t, 1 + t \rangle.$$

2. **[10 points total]** Consider the points $A(0,0,0)$ and $B(0,0,1)$.

(a) **[5 points]** Write an equation that describes the sphere S consisting of points P whose distance to B is twice the distance to A.

We seek to find the collection of all points $P = (x, y, z)$ whose distance to B is twice that to A. To do this, we have the equation:

$$2 \cdot d(P, A) = d(P, B).$$

where $d(\,\cdot\,,\,\cdot\,)$ denotes the distance function.

$$\begin{aligned} 2\sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2} &= \sqrt{(x-0)^2 + (y-0)^2 + (z-1)^2} \\ 2\sqrt{x^2 + y^2 + z^2} &= \sqrt{x^2 + y^2 + (z-1)^2} \end{aligned}$$

We now square both sides, combine like terms, and complete the square to put the sphere into a familiar form.

$$\begin{aligned} 4(x^2 + y^2 + z^2) &= x^2 + y^2 + (z-1)^2 \\ 3x^2 + 3y^2 + 3z^2 + 2z &= 1 \\ x^2 + y^2 + (z^2 + \tfrac{2}{3}z) &= \tfrac{1}{3} \\ x^2 + y^2 + (z + \tfrac{1}{3})^2 &= \tfrac{1}{3} + \tfrac{1}{9} \\ x^2 + y^2 + (z + \tfrac{1}{3})^2 &= \tfrac{4}{9} \end{aligned}$$

This final equation is the standard form of the sphere S that we seek.

(b) **[[2 points]** What is the center of the sphere S? From our equation for S, we read off: the center of the sphere is the point $(0, 0, -\frac{1}{3})$.

(c) **[2 points]** Is the point $(0, 0, 1/3)$ on the the sphere S? To check this, we find the distance from the point $(0, 0, \frac{1}{3})$ to both A and B

$$\begin{aligned} d((0,0,0), (0,0,\tfrac{1}{3})) &= \tfrac{1}{3} \\ d((0,0,1), (0,0,\tfrac{1}{3})) &= \tfrac{2}{3} \end{aligned}$$

Thus the point $(0, 0, \frac{1}{3})$ is indeed on the sphere (since it satisfies the condition that defines the sphere). We could have also plugged it into the equation for the sphere we found in part (a) and checked that it satisfied that equation.

(d) **[1 points]** What is the radius of the sphere S? Again, this can be read off from the equation of the sphere that we found in part (a): the radius of the sphere is $\sqrt{\frac{4}{9}} = \frac{2}{3}$.

3. **[10 points]** Find the curvature $\kappa(t)$ of the curve

$$\overrightarrow{\mathbf{r}}(t) = \langle 1, \sin t, \cos t \rangle.$$

We know the formula for κ, so we compute the requisite parts and simplify.

$$\kappa(t) := \frac{\left| \overrightarrow{\mathbf{r}'(t)} \times \overrightarrow{\mathbf{r}''(t)} \right|}{\left| \overrightarrow{\mathbf{r}'(t)} \right|^3}.$$

From the formula for $\mathbf{r}$, we can compute $\mathbf{r}'$, $\mathbf{r}''$:

$$\overrightarrow{\mathbf{r}'(t)} = \langle 0, \cos t, -\sin t \rangle$$
$$\overrightarrow{\mathbf{r}''(t)} = \langle 0, -\sin t, -\cos t \rangle$$

This yields the cross product:

$$\overrightarrow{\mathbf{r}'(t)} \times \overrightarrow{\mathbf{r}''(t)} = \langle -\cos^2 t - \sin^2 t, 0, 0 \rangle = \langle -1, 0, 0 \rangle$$

Finally, computing magnitudes,

$$\left| \overrightarrow{\mathbf{r}'(t)} \times \overrightarrow{\mathbf{r}''(t)} \right| = \sqrt{(-1)^2 + 0^2 + 0^2} = 1, \quad \left| \overrightarrow{\mathbf{r}'(t)} \right|^3 = \sqrt{\cos^2 t + \sin^2 t} = 1.$$

Hence $\kappa(t) = \frac{1}{1} = 1$.

4. **[5 points each part]**

(a) Find a vector function that describes the intersection of the surfaces $x = 2\sin z$ and $x^2 + y^2 = 4$.

Since $x = 2\sin z$, we substitute this into the other formula, giving

$$\begin{aligned} (2\sin z)^2 + y^2 &= 4 \\ 4\sin^2 z + y^2 &= 4 \\ y &= \sqrt{4 - 4\sin^2 z} \\ y &= 2\sqrt{1 - \sin^2 z} \\ y &= 2\cos z \end{aligned}$$

Thus one possible parameterization satisfies $x = 2\sin z$ and $y = 2\cos z$. Because this places no constraint on z, we choose $z = t$. This yields the parameterization:

$$\begin{aligned} x &= 2\sin t \\ y &= 2\cos t \\ z &= t \end{aligned}$$

In other terms,

$$\overrightarrow{\mathbf{r}(t)} = \langle 2\sin t, 2\cos t, t\rangle.$$

(b) Compute the curvature of this curve. Hint: it does not matter which point on the curve you pick!

Again, we have a handy formula to help us out. The curvature is given by

$$\kappa(t) := \frac{\left|\overrightarrow{\mathbf{r}'(t)} \times \overrightarrow{\mathbf{r}''(t)}\right|}{\left|\overrightarrow{\mathbf{r}'(t)}\right|^3}.$$

Thus we need simply to compute the first and second derivatives of out function $\mathbf{r}(t)$ and plug into the formula.

$$\begin{aligned} \overrightarrow{\mathbf{r}'(t)} &= \langle 2\cos t, -2\sin t, 1\rangle \\ \overrightarrow{\mathbf{r}''(t)} &= \langle -2\sin t, -2\cos t, 0\rangle \\ \overrightarrow{\mathbf{r}'(t) \times \mathbf{r}''(t)} &= \langle 2\cos t, -2\sin t, -4\cos^2 t - 4\sin^2 t\rangle \\ &= \langle 2\cos t, -2\sin t, -4\rangle \end{aligned}$$

Computing magnitudes,

$$\begin{aligned} \left|\overrightarrow{\mathbf{r}'(t)} \times \overrightarrow{\mathbf{r}''(t)}\right| &= \sqrt{4\cos^2 t + 4\sin^2 t + 16} = \sqrt{20} \\ \left|\overrightarrow{\mathbf{r}'(t)}\right| &= \sqrt{4\cos^2 t + 4\sin^2 t + 1} = \sqrt{5} \end{aligned}$$

Hence

$$\kappa(t) = \frac{\sqrt{20}}{\sqrt{5}^3} = \frac{2\sqrt{5}}{5\sqrt{5}} = \frac{2}{5}.$$

Since the above expression does not depend on t, we see that the curvature is constant along the curve.

5. **[10 points]** Find the plane containing the line $L = \langle 3+t, 5-t, 7+2t\rangle$ and the point $A(0,0,0)$.

Let $B = L(0) = (3,5,7)$, and $C = L(1) = (4,4,9)$. It follows that $\overrightarrow{AB} = \langle 3,5,7\rangle$ and $\overrightarrow{AC} = \langle 4,4,9\rangle$. The normal vector to the plane is

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 5 & 7 \\ 4 & 4 & 9 \end{vmatrix} = 17\mathbf{i} + \mathbf{j} - 8\mathbf{k}.$$

Since the plane contains A, we conclude that (one of) the equations of the place is

$$17x + y - 8z = 0.$$