Math 126, Section D - Spring 2014
Midterm I
April 24, 2014

Name: ____________________________________

Student ID Number: ______________

Section: DA 11:30-12:20 by Hailun ☐ DB 12:30-1:20 by Hailun ☐
DC 11:30-12:20 by Bo Peter ☐ DD 12:30-1:20 by Bo Peter ☐

exercise	**possible**	**score**
1	11	
2	11	
3	10	
4	10	
5	8	
total	50	

- Check that this booklet has all the exercises indicated above.
- TURN OFF YOUR CELL PHONE.
- Write your name and your student ID.
- This is a 50 minute test.
- You may use a scientific calculator and one 8.5×11 inch sheet of (two-sided) handwritten notes. All other electronic devices (including graphing calculators) are forbidden.
- Unless otherwise indicated, your answers should be exact instead of decimal approximations. For example $\frac{\pi}{4}$ is an exact answer and is preferable to its decimal approximation 0.78.
- Unless otherwise indicated, show your work and justify all your answers. Box your $\boxed{\text{final answer}}$.

$\boxed{\text{Sample solution}}$

Exercise 1 (7+4=11 points).

Consider the points $A = (2,7,1)$, $B = (5,3,1)$ and $C = (1,0,2)$.

(a) What is the area of the triangle that is formed by ABC?

Solution: We have $\overrightarrow{AB} = (3,-4,0)$, $\overrightarrow{AC} = (-1,-7,1)$ and

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -4 & 0 \\ -1 & -7 & 1 \end{vmatrix} = (-4,-3,-21-4) = (-4,-3,-25)$$

Then the area of the triangle is half the area of the parallelogram spanned by $\overrightarrow{AB} \times \overrightarrow{AC}$, thus

$$\boxed{\frac{1}{2}|(-4,-3,-25)| = \frac{1}{2}\sqrt{650} = \frac{5}{2}\sqrt{26}} \quad (\approx 12.7475)$$

(b) For the same triangle, what is the angle at corner A, rounded to the nearest degree?

Solution: We know that for this angle θ, we have

$$\cos(\theta) = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{|\overrightarrow{AB}| \cdot |\overrightarrow{AC}|} = \frac{-3+28}{\sqrt{9+16} \cdot \sqrt{1+49+1}} = \frac{25}{\sqrt{25} \cdot \sqrt{51}} = \frac{5}{\sqrt{51}} \approx 0.7001$$

and $\theta = \cos^{-1}(\frac{5}{\sqrt{51}}) \approx 0.7952 \approx 45.56^\circ$, which we can round to $\boxed{46^\circ}$.

Exercise 2 (6 + 5 = 11 points).

a) Find an equation of the form $Ax+By+Cz=D$ that describes the plane that contains the points $P=(5,2,1), Q=(4,2,5)$ and $R=(8,3,1)$.

Solution: We have $\overrightarrow{PQ}=(-1,0,4)$ and $\overrightarrow{PR}=(3,1,0)$. Then

$$\overrightarrow{PQ}\times\overrightarrow{PR}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 0 & 4 \\ 3 & 1 & 0 \end{vmatrix}=(-4,12,-1)=:\vec{n}$$

is the normal vector. Next, $P\cdot\vec{n}=5\cdot(-4)+2\cdot 12+1\cdot(-1)=3$. Thus the plane is

$$\boxed{-4x+12y-z=3}$$

b) The plane from above intersects the xz-plane in a line. Give the parametric equations of that line.

Solution: The xz plane consists of all points of the form $(x,0,z)$, hence they need to satisfy $-4x-z=3$.

We can take any two points in the intersection, for example $(0,0,-3)$ and $(-\frac{3}{4},0,0)$.

Then (one) direction vector of the line is $(\frac{3}{4},0,-3)$. Hence the line is

$$\vec{r}(t)=(0,0,-3)+t(\frac{3}{4},0,-3)$$

or in parametric equations

$$\boxed{x=\frac{3}{4}t, \quad y=0, \quad z=-3-3t}$$

Exercise 3 (10 points).

For the curve $\vec{r}(t) = (3\sin(2t), \cos(4t))$, find the tangent line at $t = \frac{\pi}{8}$ and give its parametric equations. What is the slope of this tangent line?

Solution: We have

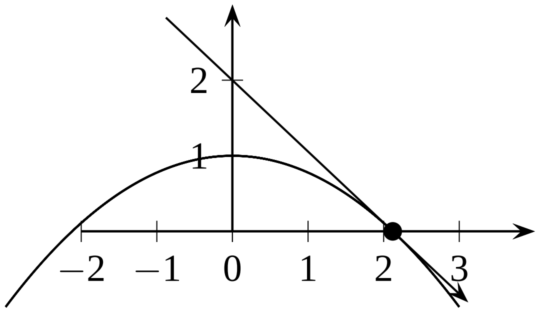
$$\begin{aligned}
\vec{r}\,'(t) &= (6\cos(2t), -4\sin(4t)) \\
\vec{r}(\frac{\pi}{8}) &= (3\sin(\frac{\pi}{4}), \cos(\frac{\pi}{2})) = (\frac{3}{\sqrt{2}}, 0) \approx (2.1213, 0) \\
\vec{r}'(\frac{\pi}{8}) &= (6\cos(\frac{\pi}{4}), -4\sin(\frac{\pi}{2})) = (\frac{6}{\sqrt{2}}, -4)
\end{aligned}$$

The canonical parametric equation of the tangent line is

$$\boxed{x = \frac{3}{\sqrt{2}} + t\cdot\frac{6}{\sqrt{2}},\ y = -4t}$$

The slope is

$$\boxed{\frac{y'(\frac{\pi}{8})}{x'(\frac{\pi}{8})} = \frac{-4}{6/\sqrt{2}} = -\frac{2}{3}\sqrt{2}} \quad (\approx -0.9428)$$



Exercise 4 (10 points).

Compute the curvature $\kappa(t)$ for the curve $\vec{r}(t) = (t,t,t^2)$.

Solution: We compute

$$\begin{aligned}
\vec{r}\,'(t) &= (1,1,2t) \\
|\vec{r}'(t)| = \sqrt{1+1+(2t)^2} &= \sqrt{2+4t^2} \\
\vec{r}''(t) &= (0,0,2) \\
\vec{r}'(t) \times \vec{r}''(t) &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 2t \\ 0 & 0 & 2 \end{vmatrix} = (2,-2,0) \\
|\vec{r}'(t) \times \vec{r}''(t)| &= \sqrt{2^2+2^2} = \sqrt{8}
\end{aligned}$$

Using the formula from the lecture

$$\kappa(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3} = \frac{\sqrt{8}}{(2+4t^2)^{3/2}}$$

Exercise 5 (8 points).

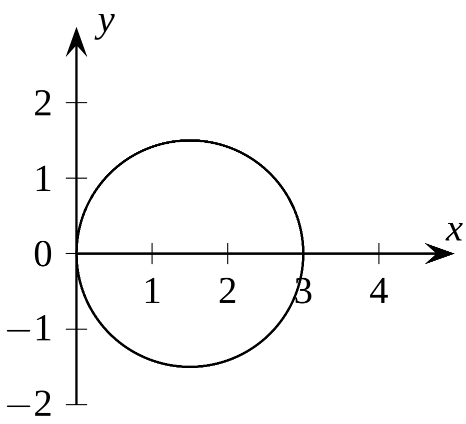
Match each polar equation to the correct curve (no justification needed).

1) $r = 1 + 3\cos(\theta)$ belongs to curve ☐ **Solution: B**

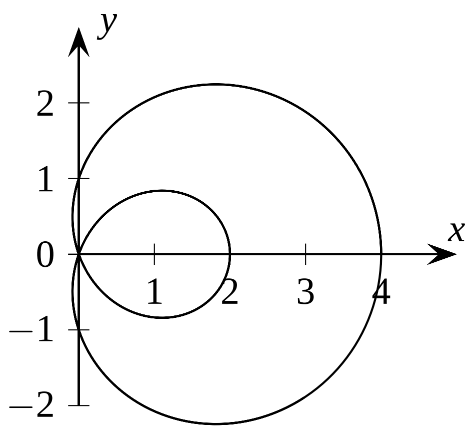
2) $r = 3\cos(2\theta)$ belongs to curve ☐ **Solution: C**

3) $r = 3\cos(\theta)$ belongs to curve ☐ **Solution: A**

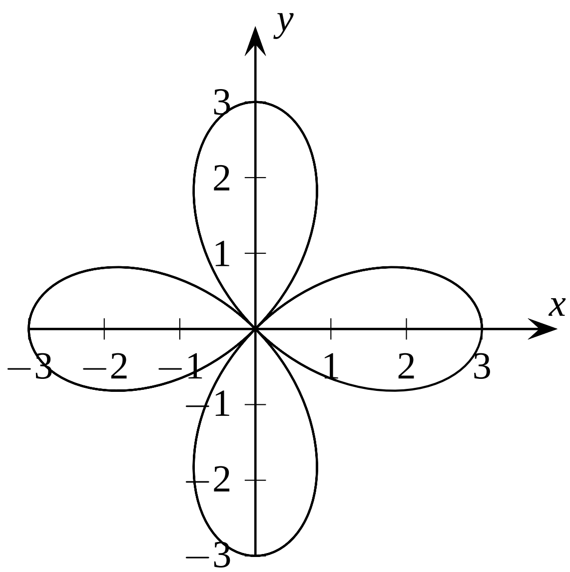
4) $r = 3\sin(\theta)$ belongs to curve ☐ **Solution: D**



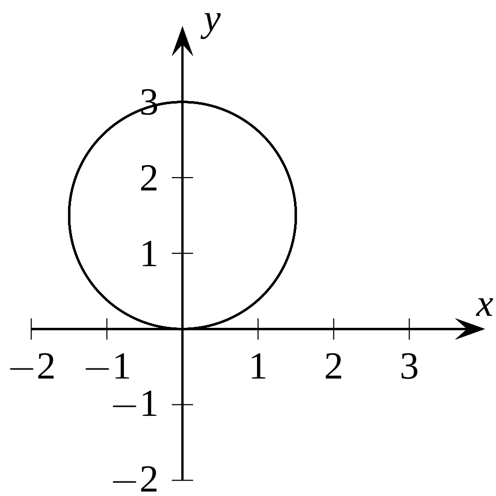
curve **A**



curve **B**



curve **C**



curve **D**