1. (8 points) Let \( r(t) = t^2 \mathbf{i} + t\sqrt{t-1} \mathbf{j} + t \sin \pi t \mathbf{k} \). Calculate the integral \( \int_1^2 r(t) \, dt \). Give your answer in exact form.

\[
\int_1^2 t^2 \, dt = \frac{7}{3}
\]

\[
\int_1^2 t\sqrt{t-1} = \int_0^1 (u+1)\sqrt{u} \, du \quad \text{(let} \ u = t - 1 \ \text{and} \ du = dt) \]

\[
= \left[ \frac{2}{5} u^{5/2} + \frac{2}{3} u^{3/2} \right]_0^1 \]

\[
= \frac{16}{15}
\]

\[
\int_1^2 t \sin \pi t \, dt = -\frac{1}{\pi} t \cos \pi t + \frac{1}{\pi^2} \sin \pi t \bigg|_1^2 \quad \text{(integration-by-parts)}
\]

\[
= -\frac{3}{\pi}
\]

\[
\int_1^2 r(t) \, dt = \left\langle \frac{7}{3}, \frac{16}{15}, -\frac{3}{\pi} \right\rangle
\]

2. (8 points) Consider the curve in \( \mathbb{R}^2 \) with parametric equations \( x = 1 + t^2, \quad y = 3t - t^3 \).

For which values of \( t \) is the curve concave upward?

\[
\frac{dx}{dt} = 2t
\]

\[
\frac{dy}{dt} = 3 - 3t^2
\]

\[
\frac{dy}{dx} = \frac{3}{2} \left( \frac{1}{t} - t \right)
\]

\[
\frac{d}{dt} \frac{dy}{dx} = -\frac{3}{2} \left( \frac{1}{t^2} + 1 \right)
\]

\[
\frac{d^2 y}{dx^2} = -\frac{3}{4} \left( 1 + t^3 + \frac{1}{t} \right) = -\frac{3}{4} \left( 1 + t^2 \right)
\]

\[
\frac{d^2 y}{dx^2} > 0 \text{ when} \ t < 0.
\]
(9 points) Compute the distance from the point \( (2, 4, 3) \) to the line of intersection of the two planes \( x + y = 2 \) and \( y + z = 3 \).

*The direction vector of the line of intersection is* \( \mathbf{v} = \langle 1, 1, 0 \rangle \times \langle 0, 1, 1 \rangle = \langle 1, -1, 1 \rangle \)

*Let \( P \) be \( (2, 4, 3) \) and note that \( Q(0, 2, 1) \) is on the intersection of the planes.*

*Let \( \mathbf{u} \) be the vector from \( Q \) to \( P \). Then \( \mathbf{u} = \langle 2, 2, 2 \rangle \)

*The distance is the magnitude of \( \mathbf{u} - \text{proj}_v \mathbf{u} \).*

\[
\text{proj}_v \mathbf{u} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{v}||^2} \right) \mathbf{v} = \frac{2}{3} \langle 1, -1, 1 \rangle
\]

\[
\mathbf{u} - \text{proj}_v \mathbf{u} = \frac{4}{3} \langle 1, 2, 1 \rangle
\]

*The distance is* \( \left| \frac{4}{3} \langle 1, 2, 1 \rangle \right| = \frac{4}{3} \sqrt{6} \approx 3.266. \)
4 (8 points) Find an equation of the plane that passes through the origin and contains the line with symmetric equations \( x - 1 = 2 - y = \frac{z + 1}{4} \).

In parametric form, the line is:
\[
\begin{align*}
x &= t + 1 \\
y &= -t + 2 \\
z &= 4t - 1
\end{align*}
\]
so it has direction vector \( \mathbf{u} = \mathbf{i} - \mathbf{j} + 4\mathbf{k} \) and passes through the point \( P(1, 2, -1) \).

Let \( \mathbf{v} \) be the vector from the origin to \( P \). Then \( \mathbf{v} \) lies in the plane.

Thus \( \mathbf{u} \times \mathbf{v} = -7\mathbf{i} + 5\mathbf{j} + 3\mathbf{k} \) is normal to the plane.

The equation of the plane is \(-7x + 5y + 3z = 0\).

5 (8 points) Calculate the length of the curve
\[
x = \cos^3 t, \quad y = \sin^3 t
\]
where \( 0 \leq t \leq 2\pi \).

\[
\begin{align*}
\frac{dx}{dt} &= -3\cos^2 t \sin t \\
\frac{dy}{dt} &= 3\sin^2 t \cos t
\end{align*}
\]

\[
\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = |3\cos t \sin t|
\]

To avoid hassles with the absolute value, integrate from 0 to \( \pi/2 \) and multiply by 4.

\[
4 \int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = 4 \int_0^{\pi/2} 3\cos t \sin t dt
\]
\[
= 6 \sin^2 t \bigg|_0^{\pi/2}
\]
\[
= 6
\]
At what point do the curves in $\mathbb{R}^3$ intersect?

\[ r_1(t) = \langle t - 1, 3t, t^2 \rangle \quad \text{and} \quad r_2(t) = \langle t + 2, 1 - t, t^3 + 9 \rangle \]

Find their angle of intersection, correct to the nearest degree.

We must solve the equation $r_1(t) = r_2(s)$ for $s$ and $t$.

This is equivalent to the system of equations:

\begin{align*}
    t - 1 &= s + 2 \\
    3t &= 1 - s \\
    t^2 &= s^3 + 9
\end{align*}

The first pair of equations

\begin{align*}
    t - 1 &= s + 2 \\
    3t &= 1 - s
\end{align*}

has solution $t = 1$ and $s = -2$.

The point of intersection is given by $r_1(1) = r_2(-2) = (0, 3, 1)$.

Now compute the tangent vectors $\mathbf{u}$ and $\mathbf{v}$ at this point.

$r_1'(t) = \langle 1, 3, 2t \rangle$ and $\mathbf{u} = r_1'(1) = \langle 1, 3, 2 \rangle$

$r_2'(s) = \langle 1, -1, 3s^2 \rangle$ and $\mathbf{v} = r_2'(-2) = \langle 1, -1, 12 \rangle$

The cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$ is

\[ \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| |\mathbf{v}|} = \frac{22}{\sqrt{14} \sqrt{146}} \]

The angle is about $61^\circ$. 