1 (8 points) Let $\mathbf{r}(t) = t^2\,\mathbf{i} + t\sqrt{t-1}\,\mathbf{j} + t\sin\pi t\,\mathbf{k}$. Calculate the integral $\int_1^2 \mathbf{r}(t)\,dt$. Give your answer in exact form.

$$\int_1^2 t^2\,dt = \frac{7}{3}$$

$$\begin{aligned}\int_1^2 t\sqrt{t-1} &= \int_0^1 (u+1)\sqrt{u}\,du \quad \textit{(let } u = t-1 \textit{ and } du = dt\textit{)}\\ &= \frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2}\Big|_0^1\\ &= \frac{16}{15}\end{aligned}$$

$$\begin{aligned}\int_1^2 t\sin\pi t\,dt &= -\frac{1}{\pi}t\cos\pi t + \frac{1}{\pi^2}\sin\pi t\Big|_1^2 \quad \textit{(integration-by-parts)}\\ &= -\frac{3}{\pi}\end{aligned}$$

$$\int_1^2 \mathbf{r}(t)\,dt = \left\langle \frac{7}{3}, \frac{16}{15}, -\frac{3}{\pi}\right\rangle$$

2 (8 points) Consider the curve in $\mathbf{R}^2$ with parametric equations $x = 1 + t^2, \quad y = 3t - t^3$. For which values of t is the curve concave upward?

$$\frac{dx}{dt} = 2t$$

$$\frac{dy}{dt} = 3 - 3t^2$$

$$\frac{dy}{dx} = \frac{3}{2}\left(\frac{1}{t} - t\right)$$

$$\frac{d}{dt}\frac{dy}{dx} = -\frac{3}{2}\left(\frac{1}{t^2} + 1\right)$$

$$\frac{d^2y}{dx^2} = -\frac{3}{4}\left(\frac{1}{t^3} + \frac{1}{t}\right) = -\frac{3}{4}\left(\frac{1+t^2}{t^3}\right)$$

$$\frac{d^2y}{dx^2} > 0 \textit{ when } t < 0.$$

3 (9 points) Compute the distance from the point $(2, 4, 3)$ to the line of intersection of the two planes $x + y = 2$ and $y + z = 3$.

The direction vector of the line of intersection is $\mathbf{v} = \langle 1, 1, 0\rangle \times \langle 0, 1, 1\rangle = \langle 1, -1, 1\rangle$

Let P *be* $(2, 4, 3)$ *and note that* $Q(0, 2, 1)$ *is on the intersection of the planes.*

Let $\mathbf{u}$ *be the vector from* Q *to* P*. Then* $\mathbf{u} = \langle 2, 2, 2\rangle$

The distance is the magnitude of $\mathbf{u} - \mathrm{proj}_{\mathbf{v}}\mathbf{u}$.

$$\begin{aligned} \mathrm{proj}_{\mathbf{v}}\mathbf{u} &= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v} \\ &= \frac{2}{3}\langle 1, -1, 1\rangle \\ \mathbf{u} - \mathrm{proj}_{\mathbf{v}}\mathbf{u} &= \frac{4}{3}\langle 1, 2, 1\rangle \end{aligned}$$

The distance is $\left|\frac{4}{3}\langle 1, 2, 1\rangle\right| = \frac{4}{3}\sqrt{6} \approx 3.266.$

4 (8 points) Find an equation of the plane that passes through the origin and contains the line with symmetric equations $x - 1 = 2 - y = \dfrac{z+1}{4}$.

In parametric form, the line is

$x = t + 1$

$y = -t + 2$

$z = 4t - 1$

so it has direction vector $\mathbf{u} = \mathbf{i} - \mathbf{j} + 4\mathbf{k}$ *and passes through the point* $P(1, 2, -1)$.

Let $\mathbf{v}$ *be the vector from the origin to* P. *Then* $\mathbf{v}$ *lies in the plane.*

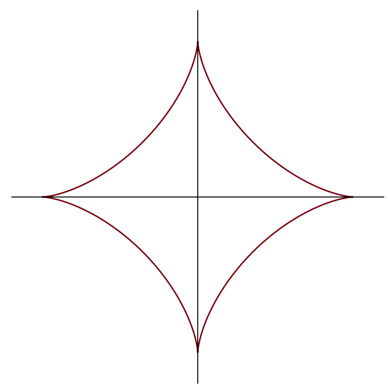
Thus $\mathbf{u} \times \mathbf{v} = -7\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}$ *is normal to the plane.*

The equation of the plane is $-7x + 5y + 3z = 0$.

5 (8 points) Calculate the length of the curve

$$x = \cos^3 t, \ \ y = \sin^3 t$$

where $0 \le t \le 2\pi$.



$$\frac{dx}{dt} = -3\cos^2 t \sin t$$

$$\frac{dy}{dt} = 3\sin^2 t \cos t$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = |3\cos t \sin t|$$

To avoid hassles with the absolute value, integrate from 0 *to* $\pi/2$ *and multiply by* 4.

$$\begin{aligned} 4\int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt &= 4\int_0^{\pi/2} 3\cos t \sin t\, dt \\ &= 6\sin^2 t\Big|_0^{\pi/2} \\ &= 6 \end{aligned}$$

6 (9 points) At what point do the curves in $\mathbf{R}^3$ intersect?

$$\begin{aligned} \mathbf{r}_1(t) &= \left\langle t-1, 3t, t^2 \right\rangle \quad \text{and} \\ \mathbf{r}_2(t) &= \left\langle t+2, 1-t, t^3+9 \right\rangle \end{aligned}$$

Find their angle of intersection, correct to the nearest degree.

We must solve the equation $\mathbf{r}_1(t) = \mathbf{r}_2(s)$ *for* s *and* t*.*
This is equivalent to the system of equations

$$\begin{aligned} t-1 &= s+2 \\ 3t &= 1-s \\ t^2 &= s^3+9 \end{aligned}$$

The first pair of equations

$$\begin{aligned} t-1 &= s+2 \\ 3t &= 1-s \end{aligned}$$

has solution $t = 1$ *and* $s = -2$*.*
The point of intersection is given by $\mathbf{r}_1(1) = \mathbf{r}_2(-2) = \langle 0, 3, 1 \rangle$*.*

Now compute the tangent vectors $\mathbf{u}$ *and* $\mathbf{v}$ *at this point.*
$\mathbf{r}_1'(t) = \langle 1, 3, 2t \rangle$ *and* $\mathbf{u} = \mathbf{r}_1'(1) = \langle 1, 3, 2 \rangle$
$\mathbf{r}_2'(s) = \langle 1, -1, 3s^2 \rangle$ *and* $\mathbf{v} = \mathbf{r}_2'(-2) = \langle 1, -1, 12 \rangle$

The cosine of the angle between $\mathbf{u}$ *and* $\mathbf{v}$ *is*
$$\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{22}{\sqrt{14}\sqrt{146}}$$

The angle is about 61°*.*