1 (7 points) Let $\mathbf{r}(t) = \frac{3}{1+t^2}\,\mathbf{i} + \frac{2t}{1+t^2}\,\mathbf{j}$. Calculate the integral $\int_0^1 \mathbf{r}(t)\,dt$. Give your answer in exact form.

$$\begin{aligned}\int_0^1 \mathbf{r}(t)\,dt &= 3\tan^{-1} t\,\mathbf{i} + \ln\left(1+t^2\right)\mathbf{j}\Big|_0^1 \\ &= \frac{3\pi}{4}\,\mathbf{i} + \ln 2\,\mathbf{j}\end{aligned}$$

2 (8 points) Consider the curve in $\mathbf{R}^2$ with parametric equations $x = 4t^2+t+1$, $y = t^4+2t$. Give the coordinates of the points on the curve where the tangent line has slope 2.

$\frac{dy}{dt} = 4t^3 + 2$ *and* $\frac{dx}{dt} = 8t+1$

We need to solve $\frac{dy}{dx} = 2$

Since $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$, *we can solve* $\frac{dy}{dt} = 2\frac{dx}{dt}$.

$$\begin{aligned}\frac{dy}{dt} &= 2\frac{dx}{dt} \\ 4t^3+2 &= 2\cdot(8t+1) \\ 4t^3 - 16t &= 0 \\ t &= 0, 2, -2\end{aligned}$$

$t = 0$ *gives the point* $(1,0)$.

$t = 2$ *gives the point* $(19,20)$

$t = -2$ *gives the point* $(15,12)$

3 (10 points) Consider the curves $\mathbf{r}_1(t) = \langle t+1, t^2+3, 3t+1\rangle$ and $\mathbf{r}_2(s) = \langle s+4, s^2, -2s\rangle$.

(a) (5 points) At what point do the curves intersect?

Solve the linear system

$$\begin{aligned} t+1 &= s+4 \\ 3t+1 &= -2s \end{aligned}$$

to get $t = 1,\ s = -2$.
Check that the y-coordinates work: $t^2 + 3 = 1^2 + 3 = 4$ *and* $s^2 = (-2)^2 = 4$.
The curves intersect at $(2, 4, 4)$

(b) (5 points) Find the (acute) angle of intersection, correct to the nearest degree.

We calculate the angle θ *between* $\mathbf{r}_1'(1)$ *and* $\mathbf{r}_2'(-2)$.
$\mathbf{r}_1'(t) = \langle 1, 2t, 3\rangle$ *so* $\mathbf{r}_1'(1) = \langle 1, 2, 3\rangle$.
$\mathbf{r}_2'(t) = \langle 1, 2s, -2\rangle$ *so* $\mathbf{r}_2'(-2) = \langle 1, -4, -2\rangle$.

$$\begin{aligned} \cos\theta &= \frac{\mathbf{r}_1'(1)\cdot\mathbf{r}_2'(-2)}{|\mathbf{r}_1'(1)|\,|\mathbf{r}_2'(-2)|} \\ &= -\frac{13}{\sqrt{14}\sqrt{21}} \end{aligned}$$

This gives $\theta = 139°$. *The acute angle is* $41°$.

4 (7 points) Calculate the area of the triangle in $\mathbf{R}^3$ with vertices $(-1,1,1)$, $(1,1,2)$ and $(-1,4,3)$.

Let $A=(-1,1,1)$*,* $B=(1,1,2)$ *and* $C=(-1,4,3)$*.*

The area of the triangle is $\frac{1}{2}|\vec{AB}\times\vec{AC}|$*.*

$\vec{AB}=\langle 2,0,1\rangle \quad \vec{AC}=\langle 0,3,2\rangle$

$\vec{AB}\times\vec{AC}=\langle -3,-4,6\rangle$

$|\vec{AB}\times\vec{AC}|=\sqrt{61}$

The area of the triangle is $\frac{1}{2}\sqrt{61}\approx 3.9$

5 (8 points) Let ℓ be the line $\mathbf{R}^3$ that passes through the points $(1,2,3)$ and $(4,1,-1)$. Find the coordinates of the point where ℓ intersects the xz-plane.

The direction vector of the line is $\langle 3,-1,-4\rangle$*.*

Parametric equations for the line are

$$\begin{aligned} x &= 3t+1 \\ y &= -t+2 \\ z &= -4t+3 \end{aligned}$$

The equation of the xz*-plane is* $y=0$*.*

Substituting the parametric equations into the plane equation gives $-t+2=0$*, so* $t=2$*.*

The corresponding point on the line is $(7,0,-5)$*.*

6 (10 points) Find an equation of the plane that passes through the points $(0, -1, 1)$ and $(2, -1, 2)$ and is perpendicular to the plane $x + y = z$.

Let $A = (0, -1, 1)$ *and* $B = (2, -1, 2)$.

The vector $\vec{AB} = \langle 2, 0, 1 \rangle$ *lies in the plane we want.*

Let $\vec{N}$ *be the normal vector to* $x + y = z$. *So* $\vec{N} = \langle 1, 1, -1 \rangle$.

Since the desired plane is perpendicular to the plane $x + y = z$, *the normal vector* $\vec{N}$ *also lies in the plane we want.*

Thus $\vec{AB} \times \vec{N}$ *is perpendicular to the desired plane.*

$\vec{AB} \times \vec{N} = \langle -1, 3, 2 \rangle$.

The plane we want has the form $-x + 3y + 2z = d$. *Plugging in point* A *gives* $d = -1$.

The desired plane is $-x + 3y + 2z = -1$.