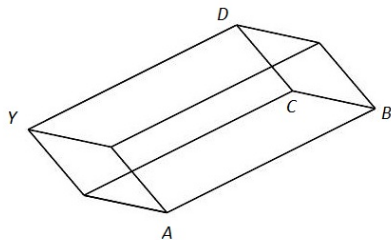


## Math 126, Sections D and F, Spring 2012, Solutions to Midterm I

1. The following questions regard the parallelepiped shown below. The figure is not to scale. The following vectors are known:  $\vec{AC} = \langle 0, 5, 2 \rangle$ ,  $\vec{AD} = \langle -14, 3, 7 \rangle$  and  $\vec{AB} = \langle 1, 3, 4 \rangle$ .



- (a) (4 points) Compute the vector  $\vec{CD}$  and the angle between  $\vec{AC}$  and  $\vec{CD}$ .

$$\vec{AC} + \vec{CD} = \vec{AD}$$

so

$$\vec{CD} = \langle -14, 3, 7 \rangle - \langle 0, 5, 2 \rangle = \langle -14, -2, 5 \rangle$$

To find the angle we compute

$$\vec{AC} \cdot \vec{CD} = \langle 0, 5, 2 \rangle \cdot \langle -14, -2, 5 \rangle = 0 - 10 + 10 = 0$$

so the angle is  $\pi/2$ .

- (b) (4 points) Compute the area of the triangle with vertices  $A$ ,  $C$  and  $D$ .  
Since the triangle is a right triangle the area is

$$A = \frac{1}{2} |\langle 0, 5, 2 \rangle| |\langle -14, -2, 5 \rangle| = \frac{\sqrt{29}\sqrt{225}}{2} = \frac{15\sqrt{29}}{2}.$$

- (c) (3 points) If the point  $A$  is at  $(2, 1, 0)$ , find the coordinates of the point  $Y$ .  
First we compute

$$\vec{BC} = \vec{AC} - \vec{AB} = \langle 0, 5, 2 \rangle - \langle 1, 3, 4 \rangle = \langle -1, 2, -2 \rangle$$

then

$$\vec{AY} = \vec{BC} + \vec{CD} = \langle -1, 2, -2 \rangle + \langle -14, -2, 5 \rangle = \langle -15, 0, 3 \rangle$$

so the point  $Y$  is at  $(-13, 1, 3)$ .

2. For the following pairs of line and plane equations, write one of SKEW, PERPENDICULAR, PARALLEL or THE SAME to finish each sentence. (2 points each. 1 point for the answer, 1 point for a brief explanation or computation.)
- (a) The planes  $x + y - z = 3$  and  $2x + 2y - 2z = 6$  are THE SAME.  
because one equation is twice the other.
  - (b) The planes  $2x - 4y + 6z = 7$  and  $-x + 2y - 3z = 9$  are PARALLEL.  
because their normal vectors are parallel but one equation is not a multiple of the other.
  - (c) The lines  $\mathbf{r}_1(t) = \langle 4, 1 + t, 3 \rangle$  and  $\mathbf{r}_2(t) = \langle t, 2t, 5t \rangle$  are SKEW.  
the direction vectors are not parallel and the lines do not intersect since  $\langle 4, 1 + s, 3 \rangle = \langle t, 2t, 5t \rangle$  is not possible.
  - (d) The lines  $\mathbf{r}_1(t) = \langle 2 - 3t, 4 + t, 7 + 2t \rangle$  and  $\mathbf{r}_2(t) = \langle 5 + 6t, 3 - 2t, 5 - 4t \rangle$  are THE SAME.  
The direction vectors are parallel and they have at least one common point.
  - (e) The plane  $4x - 7y + z = 3$  and the line  $\mathbf{r}(t) = \langle 4t, 8 - 7t, 5 + t \rangle$  are PERPENDICULAR.  
The normal of the plane and the direction vector of the line are parallel.

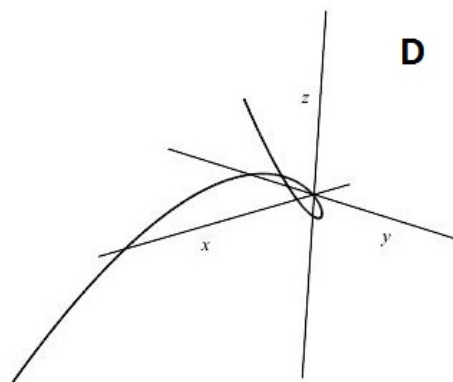
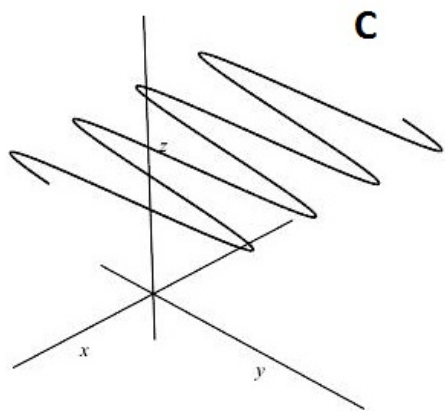
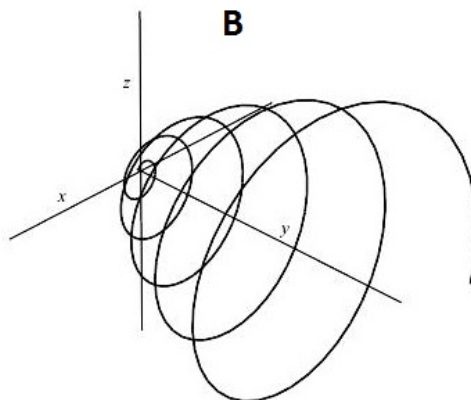
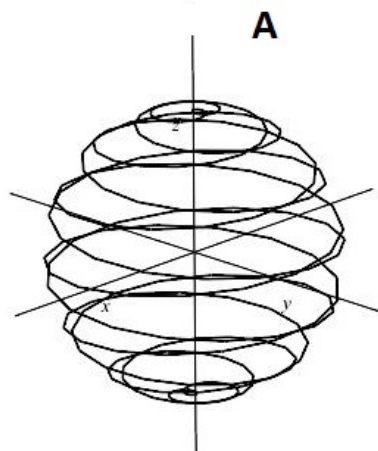
3. Answer the following questions about space curves.

- (a) (1 point each) Match the following vector functions with the space curves they represent. Think of the surfaces they are on to help you identify the graphs. Write the letter of the graph next to the equation.

$$\mathbf{r}_1(t) = \langle t, 3 + 4 \sin(t), 4 \rangle \quad \text{C} \qquad \mathbf{r}_2(t) = \langle \sin(t) \cos(14t), \sin(t) \sin(14t), \cos(t) \rangle \quad \text{A}$$

$$\mathbf{r}_3(t) = \langle t \cos(5t), t^2, t \sin(5t) \rangle \quad \text{B} \qquad \mathbf{r}_4(t) = \langle t^2 + 1, t, t^3 - 6t + 4 \rangle \quad \text{D}$$

The labels  $x$ ,  $y$  and  $z$  are next to the positive axes.



- (b) (5 points) Find parametric equations of the tangent line to the curve given by  $\mathbf{r}_1(t) = \langle t, 3 + 4 \sin(t), 4 \rangle$  at the point  $(0, 3, 4)$ .

$$\mathbf{r}'_1(t) = \langle 1, 4 \cos t, 0 \rangle$$

so the direction vector is  $\mathbf{r}'(0) = \langle 1, 4, 0 \rangle$  and the line equation is

$$\mathbf{r}(t) = \langle 0, 3, 4 \rangle + t \langle 1, 4, 0 \rangle$$

so  $x = t$ ,  $y = 3 + 4t$ ,  $z = 4$ .

4. Answer the following.

- (a) (4 points) Identify the following surface and make a sketch of it. Your picture does not have to be drawn to scale. I am only interested in seeing the shape and orientation.

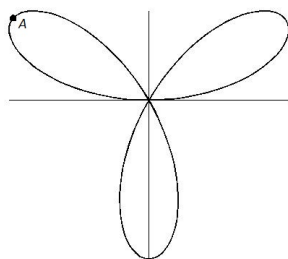
$$x^2 + 4y^2 - 24y - 4z + 20 = 0$$

When you complete the square and rearrange the equation you get

$$\left(\frac{x^2}{2}\right) + (y - 3)^2 = z + 4$$

which is an elliptic paraboloid opening up in the positive  $z$  direction whose lowest point is at  $(0, 3, -4)$ .

- (b) (6 points) Find the equation of the tangent lines to the three leaved rose  $r = \sin(3\theta)$  at the point  $A$  which is at the tip of its left petal as marked with a dot in the picture.



Compute

$$\frac{dy}{dx} = \frac{\frac{d}{d\theta}(r \sin \theta)}{\frac{d}{d\theta}(r \cos \theta)} = \frac{3 \cos 3\theta \sin \theta + \sin 3\theta \cos \theta}{2 \cos 3\theta \cos \theta - \sin 3\theta \sin \theta}$$

The tips of the petals is when  $r = \pm 1$  so when  $3\theta = \pi/2, 3\pi/2, 5\pi/2, \dots$ . If you trace the shape in the direction of increasing  $\theta$  you see that point  $A$  is when  $3\theta = 5\pi/2$  or when  $\theta = 5\pi/6$ . Plugging that we get the slope is  $\sqrt{3}$  and the point  $A$  is at  $x = r \cos \theta = \sin 3\theta \cos \theta = -\sqrt{3}/2$  and  $y = r \sin \theta = \sin 3\theta \sin \theta = 1/2$  so the tangent line is

$$y - \frac{1}{2} = \sqrt{3} \left( x + \frac{\sqrt{3}}{2} \right)$$