

1. [6 points per part] Consider the following two planes:  $x - 3y = 4$

$$P_1: 4x - 8y + z = 3 \quad P_2: x = 3y + 4$$

(a) Find the (acute) angle of intersection between  $P_1$  and  $P_2$ .

Normal vectors:  $\langle 4, -8, 1 \rangle$  and  $\langle 1, -3, 0 \rangle$

$\uparrow$                        $\uparrow$   
 $\vec{a}$                        $\vec{b}$

$$\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$$

$$\vec{a} \cdot \vec{b} = 4 + 24 + 0 = 28$$

$$|\vec{a}| = \sqrt{16 + 64 + 1} = 9$$

$$|\vec{b}| = \sqrt{1 + 9} = \sqrt{10}$$

$$28 = 9\sqrt{10} \cos \theta$$

$$\theta = \cos^{-1}\left(\frac{28}{9\sqrt{10}}\right)$$

(b) Find the line of intersection of  $P_1$  and  $P_2$ .

Write your answer in symmetric form.

Let's let  $y = t$  (arbitrarily)

$$x = 3y + 4, \text{ so } x = 3t + 4$$

$$4x - 8y + z = 3, \text{ so } 4(3t + 4) - 8t + z = 3$$

$$\text{so } z = 3 - 4(3t + 4) + 8t = -4t - 13$$

So, in parametric form:

$$\begin{aligned} x &= 3t + 4 \\ y &= t \\ z &= -4t - 13 \end{aligned}$$

In symmetric form:

$$\frac{x - 4}{3} = \frac{y}{1} = \frac{z + 13}{-4}$$

2. [2 points each] Here are six polar equations:

(A)  $r = 1 + \cos(\theta)$

(D)  $r = \frac{\theta}{10}$

(B)  $r = \cos(\theta) + 2\sin(\theta)$

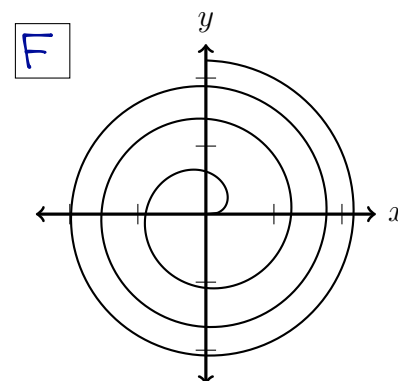
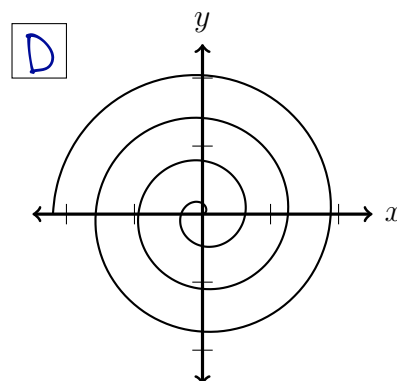
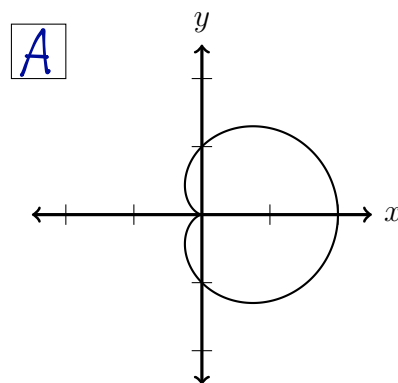
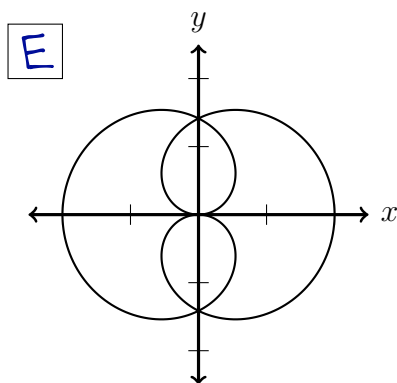
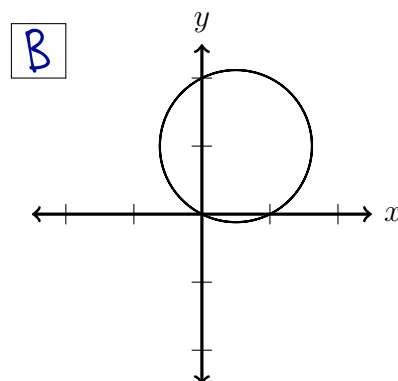
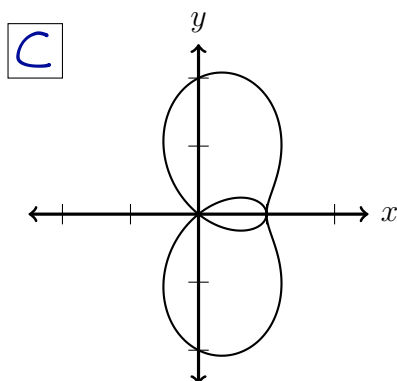
(E)  $r = 2\sin\left(\frac{\theta}{2}\right)$

(C)  $r = \cos(\theta) + 2\sin^2(\theta)$

(F)  $r = \frac{1}{2}\sqrt{\theta}$

Write the letter of each equation in the box next to its graph below.

You do not need to show any work for this problem.



The next two questions on this exam are sponsored by the vector function

$$\mathbf{r}(t) = \langle t + 1, t^2 + 6t + 8, 2t + 5 \rangle.$$

3. (a) [5 points] Write parametric equations for the line tangent to  $\mathbf{r}(t)$  at the point  $(2, 15, 7)$ .

$\vec{r}(t)$  passes through  $(2, 15, 7)$  when  $t+1=2$ , so  $t=1$ .

$\vec{r}'(t) = \langle 1, 2t+6, 2 \rangle$ , so direction vector is  $\vec{r}'(1) = \langle 1, 8, 2 \rangle$ .

Parametric equations:

$$\begin{aligned} x &= 2 + t \\ y &= 15 + 8t \\ z &= 7 + 2t \end{aligned}$$

- (b) [5 points] Find the maximum curvature of  $\mathbf{r}(t)$ .

$$\vec{r}'(t) = \langle 1, 2t+6, 2 \rangle$$

$$\vec{r}''(t) = \langle 0, 2, 0 \rangle$$

$$\vec{r}'(t) \times \vec{r}''(t) = \langle -4, 0, 2 \rangle$$

$$K = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3} = \frac{|\langle -4, 0, 2 \rangle|}{|\langle 1, 2t+6, 2 \rangle|^3} = \frac{\sqrt{20}}{(5 + (2t+6)^2)^{3/2}}$$

max when  $2t+6=0$ ,  
so  $t=-3$

$$\max K = \frac{\sqrt{20}}{5^{3/2}} = \frac{2\sqrt{5}}{5\sqrt{5}} = \boxed{\frac{2}{5}}$$

- (c) [3 points] Find the point where your maximum curvature from part (b) occurs.

$$\vec{r}(-3) = \langle -2, -1, -1 \rangle, \text{ so}$$

$$\boxed{(-2, -1, -1)}$$

4. [5 points per part] This question is again brought to you by  $\mathbf{r}(t) = \langle t+1, t^2+6t+8, 2t+5 \rangle$ .

(a) List all intersections of  $\mathbf{r}(t)$  with the plane  $y = 0$ .

$$\begin{aligned} t^2 + 6t + 8 &= 0 & \mathbf{r}(-2) &= \langle -1, 0, 1 \rangle \\ (t+2)(t+4) &= 0 & \mathbf{r}(-4) &= \langle -3, 0, -3 \rangle \\ t &= -2 \text{ or } t = -4 \end{aligned}$$

$$\text{so } \langle -1, 0, 1 \rangle \text{ and } \langle -3, 0, -3 \rangle$$

(b)  $\mathbf{r}(t)$  lies within a plane. Give the equation for that plane.

$$x = t+1, \text{ and } z = 2t+5 = 2(t+1) + 3,$$

$$\text{so } z = 2x + 3$$

(or you can find 3 points on the curve...)

(c) Find *another* surface (not a plane!) containing  $\mathbf{r}(t)$ .

Write an equation for that surface, and give its name.

Many possible answers. One is:

$$\begin{aligned} z^2 - x^2 &= (2t+5)^2 - (t+1)^2 = (4t^2 + 20t + 25) - (t^2 + 2t + 1) \\ &= 3t^2 + 18t + 24 = 3(t^2 + 6t + 8) = 3y \end{aligned}$$

$$\text{So } z^2 - x^2 = 3y \text{ a hyperbolic paraboloid.}$$

5. [8 points] I have two secret vectors  $\mathbf{u}$  and  $\mathbf{v}$ . Here are some facts:

- $\mathbf{u} + \mathbf{v} = \langle 1, 2, 3 \rangle$
- $\text{proj}_{\mathbf{u}} \mathbf{v} = \langle 4, 3, 0 \rangle$

What are  $\mathbf{u}$  and  $\mathbf{v}$ ?

Note:  $\text{proj}_{\mathbf{u}} \mathbf{v} + \mathbf{u} = \text{proj}_{\mathbf{u}} (\mathbf{u} + \mathbf{v})$

$\mathbf{u}$  points in the direction  $\langle 4, 3, 0 \rangle$  so

$$\begin{aligned} \text{proj}_{\mathbf{u}} (\mathbf{u} + \mathbf{v}) &= \text{proj}_{\langle 4, 3, 0 \rangle} \langle 1, 2, 3 \rangle \\ &= \frac{\langle 4, 3, 0 \rangle \cdot \langle 1, 2, 3 \rangle}{|\langle 4, 3, 0 \rangle|^2} \langle 4, 3, 0 \rangle \\ &= \frac{10}{25} \langle 4, 3, 0 \rangle = \langle 1.6, 1.2, 0 \rangle \end{aligned}$$

$$\text{So } \langle 4, 3, 0 \rangle + \mathbf{u} = \langle 1.6, 1.2, 0 \rangle \rightarrow \mathbf{u} = \langle -2.4, -1.8, 0 \rangle$$

$$\langle -2.4, -1.8, 0 \rangle + \mathbf{v} = \langle 1, 2, 3 \rangle \rightarrow \mathbf{v} = \langle 3.4, 3.8, 3 \rangle$$

