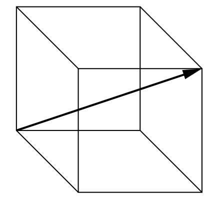

1 (8 points) Find the angle between a diagonal of a cube and one of its edges. Give your answer rounded to the nearest degree.

We may assume the length of a side of the cube is 1.

Then the diagonal is given by the vector $\mathbf{v} = \langle \mathbf{1}, \mathbf{1}, \mathbf{1} \rangle$.

The 3 *sides are given by the vectors* $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$ *and* $\mathbf{k} = \langle 0, 0, 1 \rangle$. *Each gives the same angle.*

$$\cos\theta = \frac{\mathbf{v} \cdot \mathbf{i}}{|\mathbf{v}| \cdot |\mathbf{i}|} = \frac{1}{\sqrt{3}}$$

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 55^\circ$$

2 (10 points) Let $\mathbf{r}(t) = 3t^3\mathbf{i} + 5t^2\mathbf{j}$. Compute all the points on the curve where the tangent line passes through the point $(12, 0)$.

The curve has parametric equations $x = 3t^3$, $y = 5t^2$.

The tangent line is given by an equation of the form $y - b = m(x - a)$ *where* (a, b) *is a point on the curve.*

Thus $a = 3t^3$ *and* $b = 5t^2$.

$$m = \frac{dy/dt}{dx/dt} = \frac{10t}{9t^2} = \frac{10}{9t}$$

The tangent line passes through the point $(12, 0)$ *means* $x = 12$ *and* $y = 0$.

Putting it all together gives

$$\begin{aligned} 0 - 5t^2 &= \frac{10}{9t}\left(12 - 3t^3\right) \\ -45t^3 &= 120 - 30t^3 \\ t^3 &= -8 \\ t &= -2 \end{aligned}$$

There is only one point and it has coordinates $(-24, 20)$.

3 (10 points) Compute symmetric equations for the line of intersection of the planes $2x + y - z = 2$ and $x - y - 2z = 1$. Where does this line intersect the plane $x - z = 1$?

We need a point on the line and the direction vector.

To get the point, add the 2 equations together to get $3x - 3z = 3$.

Take $z = 0$ *to get* $x = 1$. *Plug these values into* $2x + y - z = 2$ *to get* $y = 0$.

Thus the point $(1, 0, 0)$ *is on both planes.*

The direction vector is the cross product of the 2 plane normals.

$\langle 2, 1, -1\rangle \times \langle 1, -1, -2\rangle = \langle -3, 3, -3\rangle$

We can use $\langle 1, -1, 1\rangle$

The parametric equations are $x = t + 1$, $y = -t$ *and* $z = t$

The symmetric equations are $x - 1 = -y = z$

To intersect the line with the plane $x - z = 1$, *substitute the parametric equations into the plane equation.*

$$\begin{aligned} x - z &= 1 \\ (t+1) - t &= 1 \\ 1 &= 1 \end{aligned}$$

This equation is true for all values of t. *Thus the line lies in the plane* $x - z = 1$.

4 (12 points) Let $\mathbf{r}(t) = \left\langle \cos(\pi t),\ t\sin(\pi t),\ t^3 \right\rangle$.

(a) Give parametric equations for the tangent line to this curve at the point $(1, 0, -8)$.

We have the point, so we only need the direction vector. Note that $\mathbf{r}(-2) = \langle 1, 0, -8\rangle$

$\mathbf{r}'(t) = \langle -\pi\sin(\pi t), \sin(\pi t) + \pi t\cos(\pi t), 3t^2\rangle$
$\mathbf{r}'(-2) = \langle 0, -2\pi, 12\rangle$

The parametric equations are $x = 1,\ y = -2\pi t$ *and* $z = 12t - 8$

(b) Compute the curvature at the given point.

We use the equation $\kappa = \dfrac{|\mathbf{r}'(-2) \times \mathbf{r}''(-2)|}{|\mathbf{r}'(-2)|^3}$

We have $\mathbf{r}'(-2) = 2\langle 0, -\pi, 6\rangle$ *from part (a).*

$\mathbf{r}''(t) = \langle -\pi^2\cos(\pi t), \pi\cos(\pi t) + \pi\cos(\pi t) - \pi^2 t\sin(\pi t), 6t\rangle$
$\mathbf{r}''(-2) = \langle -\pi^2, 2\pi, -12\rangle$

$\mathbf{r}'(-2) \times \mathbf{r}''(-2) = -2\pi^2\langle 0, 6, \pi\rangle$
$|\mathbf{r}'(-2) \times \mathbf{r}''(-2)| = 2\pi^2\sqrt{36 + \pi^2}$

$$\kappa = \frac{2\pi^2\sqrt{36+\pi^2}}{\left(2\sqrt{36+\pi^2}\right)^3} = \frac{\pi^2}{4(36+\pi^2)} \approx 0.054$$

5 (10 points) Consider the polar curve $r = e^{2\theta}$ where $0 \leq \theta \leq 2\pi$. Find all points on the curve where the tangent line has slope 3. Give your answer in xy coordinates.

First give x and y as paramteric functions of θ.

$$\begin{aligned} x &= e^{2\theta}\cos\theta \\ y &= e^{2\theta}\sin\theta \end{aligned}$$

Compute dy/dx and set it equal to 3.

$$\begin{aligned} dx/d\theta &= 2e^{2\theta}\cos\theta - e^{2\theta}\sin\theta \\ dy/d\theta &= 2e^{2\theta}\sin\theta + e^{2\theta}\cos\theta \\ dy/dx &= \frac{2\sin\theta + \cos\theta}{2\cos\theta - \sin\theta} = 3 \end{aligned}$$

Solve for θ.

$$\begin{aligned} 2\sin\theta + \cos\theta &= 6\cos\theta - 3\sin\theta \\ 5\sin\theta &= 5\cos\theta \\ \tan\theta &= 1 \\ \theta &= \pi/4,\ 5\pi/4 \end{aligned}$$

Use the parametric equations to calculate the points.

$\left(\frac{e^{\pi/2}}{\sqrt{2}}, \frac{e^{\pi/2}}{\sqrt{2}}\right)$ *and* $\left(-\frac{e^{5\pi/2}}{\sqrt{2}}, -\frac{e^{5\pi/2}}{\sqrt{2}}\right)$

Or approximately

$(3.4, 3.4)$ *and* $(-1821.5, -1821.5)$