Math 126, Section C, Autumn 2012, Solutions to Midterm I

1. Answer the following question regarding the picture below

   We know $\vec{AC} = \langle 2, 6, 2\rangle$, $\vec{BD} = \langle 4, 0, -2\rangle$ and $A = (0, 2, -1)$.

   (a) (4 points) Compute the two vectors **u**$= \vec{AB} = \vec{DC}$ and **v**$= \vec{AD} = \vec{BC}$. From **v**+**u**$= \vec{AC} =< 2, 6, 2 >$ and **v**-**u**$= \vec{BD} =< 4, 0, -2 >$ we have 2**v**$=< 6, 6, 0 >$ and **v**$=< 3, 3, 0 >$ and **u**$=< 2, 6, 2 > - < 3.3.0 >=< -1, 3, 2 >$.

   (b) (3 points) Find the coordinates of the points B and C. From $< 2, 6, 2 >= \vec{AC}$ we get $C = (2, 8, 1)$. From $< -1, 3, 2 >= \vec{AB}$ we get $B = (-1, 5, 1)$.

   (c) (3 points) The line containing B and E is perpendicular to the line containing A to D as shown in the picture. Find the coordinates of the point E.

   $$\vec{AE} = \mathbf{proj_v u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v}$$

   $$= \frac{< -1, 3, 2 > \cdot < 3, 3, 3 >}{< 3, 3, 0 > \cdot < 3, 3, 0 >} < 3, 3, 0 >=< 1, 1, 0 >$$

   so $E = (1, 3, -1)$.

2. Given two planes

   $$P1: \quad 2x - y + z = 5$$

   and

   $$P2: \quad 3x + 2y - z = 3,$$

   (a) (6 points) Find parametric equations for the line of intersection of the two planes. Check that your line is indeed on both planes.

   The direction vector for the line is $\mathbf{n_1} \times \mathbf{2_2}$$=< 2, -1, 1 > \times < 3, 2, 1 >=< -1, 5, 7 >$.

   To compute a point on both planes we can take $x = 0$ and solve $0 - y + z = 5$ and $0 + 2y - z = 3$ to get $y = 8$ and $z = 13$ so the point is $(0, 8, 13)$.

   Therefore the line equations are

   $$x = 0 - t, y = 8 + 5t, z = 13 + 7t.$$

   (b) (3 points) Find the equation of a third plane $P3$ which contains that line and the point $P(0, 7, 2)$. The normal for the plane is **n**=**v**$\times\vec{PQ}$ where $Q = (0, 8, 13)$. So the normal vector is $< -1, 5, 7 > \times < 0, 1, 11 >=< 48, 11, -1 >$. Therefore, the plane equation is

   $$48x + 17y - z = 75.$$

   (c) (1 point) Find the line of intersection of the planes $P1$ and $P3$.
   The same as the line in part (a)

3. Answer the following.

   (a) (6 points) Match the following vector functions with the curves they trace in space. The positive z-axis points up in the graphs. Write the letter of the graph next to the corresponding vector function.

   $\mathbf{r_1}(t) = \langle t + 3, 2t - 1, -t + 4\rangle$ F $\quad$ $\mathbf{r_2}(t) = \langle 2t + 3, 2t - 1, t + 4\rangle$ E $\quad$ $\mathbf{r_3}(t) = \left\langle t\cos(t), t, \frac{t\sin(t)}{2}\right\rangle$ D

   $\mathbf{r_4}(t) = \langle t, \sin(t), 0\rangle$ B $\quad$ $\mathbf{r_5}(t) = \left\langle t + 1, 2t^2 - 5t + 1, t^3\right\rangle$ E $\quad$ $\mathbf{r_6}(t) = \left\langle \cos(t), 10t, \frac{\sin(t)}{2}\right\rangle$ F

(b) (4 points) Find the vector equation of the tangent line to $\mathbf{r}(t) = \langle t+1, 2t^2 - 5t + 1, t^3 \rangle$ at the point where $t = 2$.
Plugging in $t = 2$, the point is $(2+1, 8-10+1, 8) = (3, -1, 8)$. The drection vector is $\mathbf{r}'(2)$. We compute $\mathbf{r}'(t) =< 1, 4t-5, 3t^2 >$ so $\mathbf{r}'(2) =< 1, 3, 12 >$. Therefore the line equation is $\mathbf{r}(t) =< 3+t, -1+3t, 8+12t >$.

4. Given the equation
$$x^2 - 4y^2 + 4z^2 + 8y = 4,$$

(a) (5 points) Identify the surface and sketch it. Label your axes so I can see the orientation. Label any points you think are important, for example, if you have a sphere, label its center.
Completing the square and rearranging terms we get
$$\frac{x^2}{4} + z^2 = (y-1)^2$$
which is a double cone with the y axis running through it. The two cones meet at the point $(0, 1, 0)$.

(b) (4 points) Find the point(s) of intersection of the above surface and the line given by
$$x = 8t \qquad y = 5t + 1 \qquad z = 3 - t.$$
We solve
$$(8t)^2 - 4(5t + 1 - 1)^2 + 4(3 - t)^2 = 0$$
simplifying we get
$$-t^2 - 6t + 9 = 0$$
so $t = -3/2$ or $t = -3/4$, giving us the points $(-12, -13/2, 9/2)$ and $(-6, -11/4, 15/4)$.

(c) (1 point) Write one vector function which gives a curve on this cone. There are many answers to this question.
Here any three functions $f(t)$, $g(t)$, $h(t)$ with
$$\frac{f(t)^2}{4} + h(t)^2 = (g(t) - 1)^2$$
would give a curve on the cone. For example $\mathbf{r}(t) =< 2t, t+1, 0 >$ or $\mathbf{r}(t) =< 2t\cos t, t+1, t\sin t >$.