1 (10 points) Consider the curve given by the equation in polar coordinates

$$r = 1 + \sin\theta.$$

Find the equation of the tangent line to the curve at $\theta = \pi/6$.

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$x = r\cos\theta = (1+\sin\theta)\cos\theta = \cos\theta + \sin\theta\cos\theta$$

$$y = r\sin\theta = (1+\sin\theta)\sin\theta = \sin\theta + \sin^2\theta$$

$$\frac{dx}{d\theta} = -\sin\theta + \cos^2\theta - \sin^2\theta, \quad \frac{dy}{d\theta} = \cos\theta + 2\sin\theta\cos\theta$$

$$\frac{dx}{d\theta}\left(\frac{\pi}{6}\right) = -\frac{1}{2} + \left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^2 = -\frac{1}{2} + \frac{3}{4} - \frac{1}{4} = 0$$

the slope of the tangent line is ∞, so the tangent line is <u>vertical</u>

For $\theta = \frac{\pi}{6}$ we have $x = \cos\frac{\pi}{6} + \sin\frac{\pi}{6}\cos\frac{\pi}{6}$

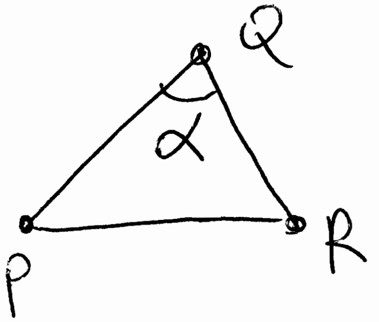
$$= \frac{\sqrt{3}}{2} + \frac{1}{2}\,\frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{4}$$

So the equation of the tangent line is

$$\boxed{x = \frac{3\sqrt{3}}{4}}$$

2 (10 points total) Three points are given: $P(0,-1,1)$, $Q(1,2,2)$, and $R(3,1,0)$.

(a) (5 points) Find the area of the triangle PQR.



$$A = \frac{1}{2}\left|\overrightarrow{QP} \times \overrightarrow{QR}\right|$$

$$\overrightarrow{QP} = \langle -1, -3, -1\rangle, \quad \overrightarrow{QR} = \langle 2, -1, -2\rangle$$

$$\overrightarrow{QP} \times \overrightarrow{QR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & -3 & -1 \\ 2 & -1 & -2 \end{vmatrix} = \langle 6-1, -(2+2), 1+6\rangle = \langle +5, -4, 7\rangle$$

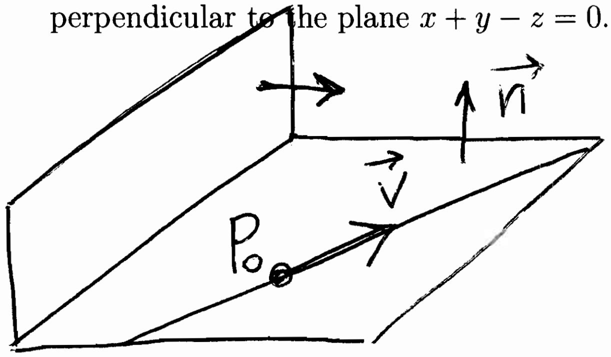
$$A = \frac{1}{2}\left|\langle 5, -4, 7\rangle\right| = \frac{1}{2}\sqrt{5^2+4^2+7^2} = \boxed{\frac{1}{2}\sqrt{90}}$$

(b) (5 points) Find the cosine of the angle of the triangle PQR at the vertex Q.

$$\cos\alpha = \frac{\overrightarrow{QP}\cdot\overrightarrow{QR}}{|\overrightarrow{QP}|\,|\overrightarrow{QR}|} = \frac{\langle -1, -3, -1\rangle \cdot \langle 2, -1, -2\rangle}{\sqrt{1+9+1}\ \sqrt{4+1+4}}$$

$$= \frac{-2+3+2}{3\sqrt{11}} = \boxed{\frac{1}{\sqrt{11}}}$$

3 (10 points) Find an equation of the plane which contains the line $\frac{x-1}{2} = \frac{y+2}{-3} = \frac{z-3}{4}$ and is perpendicular to the plane $x + y - z = 0$.



Let $\vec{n}$ be the normal vector for the plane we are looking for.

Then $\vec{n}$ must be orthogonal to $\vec{v} = \langle 2, -3, 4 \rangle$, the vector parallel to the line, and to $\langle 1, 1, -1 \rangle$, the vector normal to the other plane.

We can take $\vec{n} = \langle 2, -3, 4 \rangle \times \langle 1, 1, -1 \rangle$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -3 & 4 \\ 1 & 1 & -1 \end{vmatrix} = \langle 3-4, -(-2-4), 2+3 \rangle = \langle -1, 6, 5 \rangle$$

As a point on the plane we can take $P_0(1, -2, 3)$ which is on the line. Then we get the equation:

$$-(x-1) + 6(y+2) + 5(z-3) = 0 \quad \text{or}$$

$$\boxed{-x + 6y + 5z = 2}$$

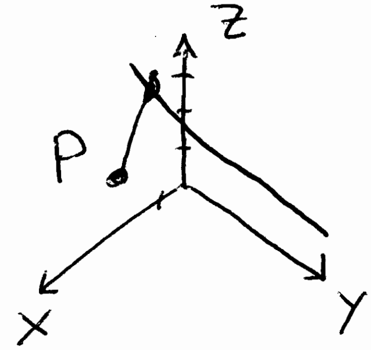
4 (10 points) Let S be the surface defined as the set of points $P(x, y, z)$ such that the distance from P to the plane $x = 2$ equals the distance from P to the line $x = 1,\ z = 3$. Find an equation for S. Simplify the equation and determine what kind of surface this is.

the distance from $P(x,y,z)$ to the plane $x=2$

is $|x-2|$

the distance from $P(x,y,z)$ to the line $x=1, z=3$

is $\sqrt{(x-1)^2+(z-3)^2}$



(since the line is parallel to the y-axis, it is the distance from (x,y,z) to $(1,y,3)$)

$$|x-2| = \sqrt{(x-1)^2+(z-3)^2}$$

Simplify: $(x-2)^2 = (x-1)^2 + (z-3)^2$

$$x^2-4x+4 = x^2-2x+1+(z-3)^2$$

$$-2x+3 = (z-3)^2$$

$$\boxed{x = \frac{3}{2} - \frac{1}{2}(z-3)^2}$$

this is a (parabolic) cylinder

5 (10 points total)

(a) (2 points) Identify the surface given by the equation $3x^2 = y^2 + z^2$ (sketch is not required).

this is a $\boxed{\text{(circular) cone}}$

(b) (8 points) Find a vector function $\vec{r}(t)$ that represents the curve of the intersection of the surfaces $3x^2 = y^2 + z^2$ and $y + 2x = 1$.

$y = 1 - 2x$, $3x^2 = (1-2x)^2 + z^2$

$3x^2 = 1 - 4x + 4x^2 + z^2$

$x^2 - 4x + 1 + z^2 = 0$

in xz-plane $(x-2)^2 + z^2 = 3$

this is a circle centered at $(2,0,0)$ of radius $\sqrt{3}$. Parametrize the circle:

$x = 2 + \sqrt{3}\cos t$, $z = \sqrt{3}\sin t$, $0 \le t \le 2\pi$

$y = 1 - 2x = 1 - 4 - 2\sqrt{3}\cos t = -3 - 2\sqrt{3}\cos t$

$$\boxed{\vec{r}(t) = \langle 2 + \sqrt{3}\cos t, -3 - 2\sqrt{3}\cos t, \sqrt{3}\sin t \rangle \quad 0 \le t \le 2\pi}$$