Math 126D **Midterm I** **Autumn 2010**

 # 1. (25 points) The points $A(1,2,3)$, $B(0,1,3)$, and $C(2,-1,-1)$ determine a triangle. **(a)** Find the interior angle of the triangle at vertex A. **(b)** What is the area of the triangle? **(c)** What is the equation for the plane containing the points A, B,and C? **(d)**What is the distance from the point $D(2,0,1)$ to the plane containing A, B, and C?

Solution. (a) Note that $\vec{AB} = -\hat{\mathbf{i}} - \hat{\mathbf{j}}$ and $\vec{AC} = \hat{\mathbf{i}} - 3\hat{\mathbf{j}} - 4\hat{\mathbf{k}}$. Let θ be the angle at vertex A. Then $\cos(\theta) = \dfrac{\vec{AB}\cdot\vec{AC}}{|\vec{AB}||\vec{AC}|} = \dfrac{2}{\sqrt{2}\sqrt{26}} = \dfrac{1}{\sqrt{13}}$. So $\boxed{\theta = \cos^{-1}(1/\sqrt{13})}$.

(b) Let $\mathbf{n} = \vec{AB}\times\vec{AC} = 4\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$. Then $\boxed{\text{Area} = \tfrac{1}{2}|\mathbf{n}| = 2\sqrt{3}}$

(c) Since $\mathbf{n}$ is normal to the plane and A is on the plane, setting $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ and $\mathbf{r}_0 = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$, in the equation $\mathbf{n}\cdot(\mathbf{r} - \mathbf{r}_0) = 0$ yields the equation $4x - 4y + 4z - 8 = 0$ or $\boxed{x - y + z - 2}$

(d) Distance $= \left|\vec{AD}\cdot\frac{\mathbf{n}}{|\mathbf{n}|}\right| = \boxed{\frac{1}{\sqrt{3}}}$

 # 2. (25 points) Let $\mathcal{P}$ be the plane given by the equation $2x+y+z=2$; and let ℓ be the line passing through the points $A(1,1,1)$ and $B(1,1,-1)$. **(a)** Find a parametric equation for ℓ. (Give your answer in vector form $\mathbf{r} = \mathbf{r}(t)$.) **(b)** Find the point Q where ℓ intersects $\mathcal{P}$. **(c)** The line ℓ intersects $\mathcal{P}$ at an angle α. Find α.

Solution: (a) $\vec{AB} = -2\hat{\mathbf{k}}$ So $\boxed{\mathbf{r}(t) = \hat{\mathbf{i}} + \hat{\mathbf{j}} + (1-2t)\hat{\mathbf{k}}}$,

(b) Since on the line ℓ we have $x = 1$, $y = 1$ and $z = 1 - 2t$, we need to solve the equation $2(1) + (1) + (1 - 2t) = 2$ for t. This gives $t = 1$. Since $\mathbf{r}(1) = \hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$, $\boxed{Q = (1,1,-1)}$.

(c) The angle θ that ℓ makes with the normal to $\mathcal{P}$ satisfies the equation $\theta + \alpha = \pi/2$. Hence $\sin(\alpha) = \cos(\theta)$. Hence, $\alpha = \sin^{-1}\left(\dfrac{\vec{AB}\cdot\mathbf{n}}{|\vec{AB}||\mathbf{n}|}\right) = \boxed{\sin^{-1}(1/\sqrt{6})}$. (Note: $\alpha = \pi/2 - \cos^{-1}(1/\sqrt{6})$ is another way to write this.)

 # 3. (25 points) A bug moves in the (x,y) plane according to the parametric equation $\mathbf{r} = (t + t^3)\,\hat{\mathbf{i}} + \sin^2(\pi t)\,\hat{\mathbf{j}}$, where t denotes time in seconds and position is measured in meters. The bug is located at the origin at time $t = 0$ and touches the x-axis again at time $t = b$, as shown in the figure below. **(a)** Give the first time <u>after</u> time $t = 0$ and <u>before</u> time $t = b$ when the velocity of the bug is parallel to the x-axis. **(b)** Express the area of the region above the x axis and below the trajectory of the bug for $0 \le t \le b$ as a definite integral. Simplify your expression, but do not attempt to evaluate the integral. **(c)** Express the distance the bug travels in the time interval $0 \le t \le b$ as a definite integral. Simplify your expression, but do not attempt to evaluate the integral.

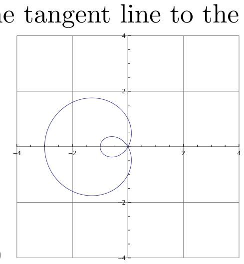
Solution. (a) The velocity is vertical when the y-coordinate reaches its maximum, which is clearly when $\sin(\pi t) = 1$, or when $\boxed{t = 1/2 \text{ sec}}$.

(b) The time b when the bug touches the x-axis occurs when $\sin(\pi t) = 0$, so $b = 1$ sec. Since $x'(t) > 0$ for $t > 0$, the area is given by

$$\text{Area} = \int_0^b y(t)x'(t)dt = \boxed{\int_0^1 \sin^2(\pi t)(1 + 3t^2)dt}.$$

(c) Similarly, distance traveled $= \displaystyle\int_0^1 \sqrt{x'(t)^2 + y'(t)^2}\,dt = \boxed{\int_0^1 \sqrt{(1+3t^2)^2 + (2\pi\sin(\pi t)\cos(\pi t))^2}\,dt}$ (Note: no points were taken off for not going further than that.)

 # 4. (25 points) Consider the curve given in polar form by the formula $r = 1 - 2\cos(\theta)$. **(a)** Carefully sketch the curve in the grid below. **(b)** Express the curve in vector form $\mathbf{r} = f(\theta)\,\hat{\mathbf{i}} + g(\theta)\,\hat{\mathbf{j}}$. **(c)** Find the parametric equation for the tangent line to the curve at the point $\theta = \pi/2$. (Do this in rectangular coordinates.)

(a)  (b) $\boxed{\mathbf{r}(t) = (1 - 2\cos(\theta))\cos(\theta)\hat{\mathbf{i}} + (1 - 2\cos(\theta))\sin(\theta)\hat{\mathbf{j}}}$

(c) $\mathbf{r}(\pi/2) = \hat{\mathbf{j}}$ and $\mathbf{r}'(\pi/2) = -\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$. So $\boxed{\langle f(t), g(t)\rangle = \langle -t, 1 + 2t\rangle}$.