1 (8 points total) Recall that $\vec{\imath}$, $\vec{\jmath}$, and $\vec{k}$ are the standard basis vectors. Give a **concrete** example of each of the following:

(a) (3 points) A **nonzero** vector $\vec{v}$ such that $\mathrm{proj}_{\vec{k}}\vec{v} = \vec{0}$.

For example, take $\vec{v} = \vec{\imath}$

$\vec{\imath} \cdot \vec{k} = 0$, so the vector projection of $\vec{v}$ to $\vec{k}$ is $\vec{0}$

(b) (5 points) A **unit** vector that is perpendicular to both $\vec{\imath} + \vec{\jmath}$ and $\vec{\jmath} - \vec{k}$. How many different solutions are there?

perpendicular vector: $(\vec{\imath} + \vec{\jmath}) \times (\vec{\jmath} - \vec{k}) = -\vec{\imath} + \vec{\jmath} + \vec{k}$

$= \langle -1, 1, 1 \rangle$

unit vector:

$$\vec{n} = \frac{\langle -1, 1, 1 \rangle}{|\langle -1, 1, 1 \rangle|} = \left\langle \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$$

2 solutions: $\vec{n}$ and $-\vec{n}$

2 (10 points) Consider the curve with the vector equation

$$\vec{r}(t) = \langle t, t^2 + 1, t^3 - 2t^2 \rangle$$

Is there a point on this curve where the tangent line is parallel to the vector $\langle 10, 40, 40 \rangle$? If so, find the point. If not, explain why.

tangent vector: $\vec{r}\,'(t) = \langle 1, 2t, 3t^2 - 4t \rangle$

parallel means there is a scalar k such that

$k \langle 1, 2t, 3t^2 - 4t \rangle = \langle 10, 40, 40 \rangle$

Solve for k: and t $\quad \boxed{k = 10}$, $2tk = 40$

$(3t^2 - 4t)k = 40$

$t = \dfrac{40}{2k} = \dfrac{40}{20} = 2$, $\boxed{t = 2}$

try if $(3t^2 - 4t)k = 40$: $(3 \cdot 2^2 - 4 \cdot 2)10 = 40$

Yes!

So the answer is $\boxed{\text{yes}}$

at the point $\boxed{\vec{r}(2) = \langle 2, 5, 0 \rangle}$

3 (10 points total) Consider two planes given by the equations $x+2y-3z=5$ and $2x-y+z=0$.

(a) (5 points) Find parametric equations of the line where the planes intersect.

Let $z=0$. Then $x+2y=5$, $2x-y=0 \Rightarrow x=1, y=2$

$P(1,2,0)$ is on the line.

$$\vec{v} = \vec{n_1} \times \vec{n_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -3 \\ 2 & -1 & 1 \end{vmatrix} = \langle -1, -7, -5 \rangle$$

$$\boxed{x = 1-t,\ y = 2-7t,\ z = -5t}$$

Alternate solution: Choose $x=t$. Then

$$\left.\begin{array}{ll} (1) & 2y-3z=5-t \\ (2) & -y+z=-2t \end{array}\right| \underset{Eq(1)+2Eq(2)}{\Rightarrow} -z = 5-5t,\quad \begin{array}{l} z=-5+5t \\ y = z+2t = -5+7t \end{array}$$

$$\boxed{x=t,\ y=-5+7t,\ z=-5+5t}$$

(b) (5 points) Find the cosine of the angle between the planes.

$$\cos\theta = \frac{|\vec{n_1} \cdot \vec{n_2}|}{|\vec{n_1}||\vec{n_2}|} = \frac{|\langle 1,2,-3\rangle \cdot \langle 2,-1,1\rangle|}{\sqrt{1+4+9}\ \sqrt{4+1+1}}$$

$$= \frac{3}{\sqrt{14}\sqrt{6}} = \boxed{\frac{3}{2\sqrt{21}}}$$

Note: the angle between planes is acute $\left(\le \frac{\pi}{2}\right)$ by definition, but we give full credit to those who have the answer $-\frac{3}{2\sqrt{21}}$

4 (12 points total) Consider the curve given by the equation in polar coordinates

$$r = 2\cos(\theta) + 4\sin(\theta).$$

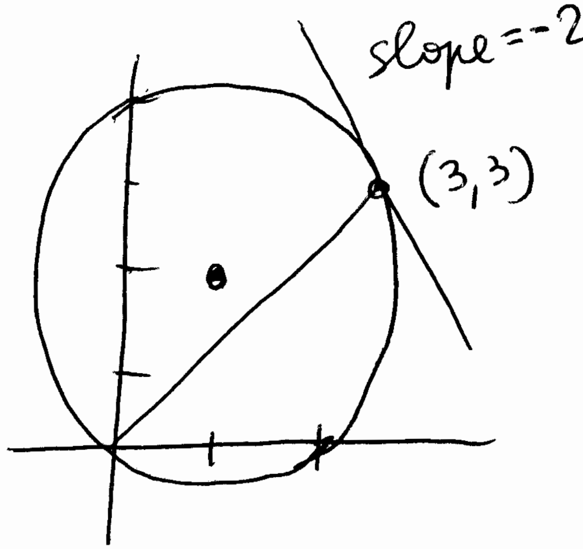
(a) (6 points) Find the Cartesian equation (non-parametric, in x and y coordinates) of the curve. Sketch the curve.

$$r^2 = 2r\cos\theta + 4r\sin\theta$$

$$x^2 + y^2 = 2x + 4y$$

$$x^2 - 2x + y^2 - 4y = 0$$

$$(x-1)^2 + (y-2)^2 = 5$$



Circle of radius $\sqrt{5}$ centered at $(1,2)$

(b) (6 points) Find the equation of the tangent line to the curve at $\theta = \pi/4$.

$$x = r\cos\theta, \quad y = r\sin\theta$$

$$dy/dx = \frac{dy/d\theta}{dx/d\theta} = \left.\frac{dr/d\theta \sin\theta + r\cos\theta}{dr/d\theta \cos\theta - r\sin\theta}\right|_{\theta=\pi/4} = \frac{\sqrt{2}\sqrt{2}/2 + \frac{3\sqrt{2}\sqrt{2}}{2}}{\frac{\sqrt{2}\sqrt{2}}{2} - \frac{3\sqrt{2}\sqrt{2}}{2}} = \boxed{-2}$$

at $\theta = \pi/4$: $\sin\theta = \cos\theta = \pi/4$, $r = \frac{6\sqrt{2}}{2} = 3\sqrt{2}$,

$$dr/d\theta = -2\sin\theta + 4\cos\theta = 2\sqrt{2}/2 = \sqrt{2}$$

$$x = 3\sqrt{2}\cdot\sqrt{2}/2 = 3, \quad y = 3\sqrt{2}\,\sqrt{2}/2 = 3$$

tangent line: $\boxed{y - 3 = -2(x-3)}$

5 (10 points total) Consider the surface defined as the set of points which are equidistant from the x-axis and from the yz-plane.

(a) (6 points) Write down the equation of the surface.

Distance from $P(x, y, z)$ to the x-axis is $\sqrt{y^2+z^2}$

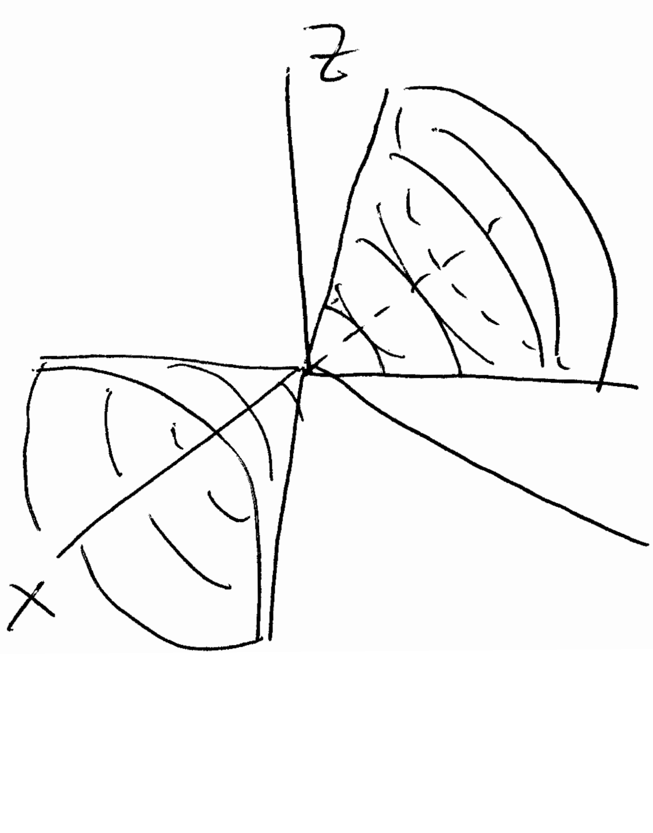
Distance from $P(x, y, z)$ to the yz-plane is $|x|$

$$|x| = \sqrt{y^2+z^2}$$

$$x^2 = y^2+z^2$$

(b) (4 points) Identify the surface.

This is a cone



(sketch was not required)