1. *(2 points each)* Each of the following multiple choice problems has one correct answer. Circle it. You do not need to show any reasoning.

   (a) If $|\mathbf{b}| = -2\,\text{comp}_{\mathbf{a}}\mathbf{b}$, then what's the angle between $\mathbf{a}$ and $\mathbf{b}$?

   (i) 30° (ii) 60° (iii) 120° (iv) 150°

   $|\vec{b}| = -2|\vec{b}|\cos\theta$

   $\cos\theta = \frac{-1}{2}$

   (b) Suppose $\overrightarrow{AB} \times \overrightarrow{AC} = \langle 2, -1, 2\rangle$. What's the area of $\triangle ABC$?

   (i) 1 (ii) 1.5 (iii) 2 (iv) 3

   $\text{area} = \frac{1}{2}|\vec{AB} \times \vec{AC}| = \frac{1}{2}\sqrt{4+1+4} = 1.5$

   (c) Which of the following points is on the line $\mathbf{r}(t) = \langle 2-t, t, -1+2t\rangle$?

   (i) $(1, 1, 1)$. (ii) $(1, 1, 0)$. (iii) $(-1, 1, 2)$. (iv) $(2, 1, 1)$.

   $\vec{r}(1) = \langle 1, 1, 1\rangle$

   (d) What's the intersection between the planes $2x - y + 3z = 3$ and $4x - 2y + 6z = 5$?

   (i) a point (ii) a line (iii) a plane (iv) nothing

   $4x - 2y + 6z = 5$

   $\div 2 \downarrow$

   $2x - y + 3z = 2.5$ this has no intersection w/ $2x - y + 3z = 3$, because $2.5 \neq 3$

   (e) What is the trace (cross section) of the surface $x^2 - 2x - y^2 + z^2 = 0$ in the plane $z = 1$?

   (i) a parabola (ii) a circle (iii) a hyperbola (iv) two lines

   $x^2 - 2x - y^2 + 1 = 0$

   $(x-1)^2 - y^2 = 0$ $(x-1) = \pm y$

2. *(12 points)* Consider the space curve

$$\mathbf{r}(t) = \langle t, 1 + \sin t, 2t - \cos t \rangle.$$

Find the equation of the plane that passes through the point $P(1, 3, -4)$ and contains the tangent line to the curve $\mathbf{r}(t)$ at $t = 0$.

Tangent vector @ $t=0$:

$\vec{r}\,'(t) = \langle 1, \cos t, 2 + \sin t \rangle$

$\vec{r}\,'(0) = \langle 1, 1, 2 \rangle$

$\vec{r}(0) = \langle 0, 1, -1 \rangle$

Vector from $(0,1,-1)$ to $(1,3,-4)$:

$\langle 1, 2, -3 \rangle$

normal vector is cross product of these:

$\vec{n} = \langle 1,1,2 \rangle \times \langle 1,2,-3 \rangle$

$= \langle -7, 5, 1 \rangle$

Plane through $(1,3,-4)$ w/ normal vec. $\langle -7,5,1 \rangle$

$-7(x-1) + 5(y-3) + (z+4) = 0$

Plane: $\boxed{-7x + 5y + z = 4}$

3. *(12 points)* Let $f(x, y) = \ln(x+1) + \tan^{-1}(y) - \cos(xy)$. Find all of the first and second partial derivatives of f.

$$f_x(x,y) = \frac{1}{x+1} + y\sin(xy)$$

$$f_y(x,y) = \frac{1}{1+y^2} + x\sin(xy)$$

$$f_{xx}(x,y) = \frac{-1}{(x+1)^2} + y^2\cos(xy)$$

$$f_{xy}(x,y) = \sin(xy) + xy\cos(xy)$$

$$f_{yy}(x,y) = \frac{-2y}{(1+y^2)^2} + x^2\cos(xy)$$

$f_x(x, y) =$ $\frac{1}{x+1} + y\sin(xy)$

$f_y(x, y) =$ $\frac{1}{1+y^2} + x\sin(xy)$

$f_{xx}(x, y) =$ $\frac{-1}{(x+1)^2} + y^2\cos(xy)$

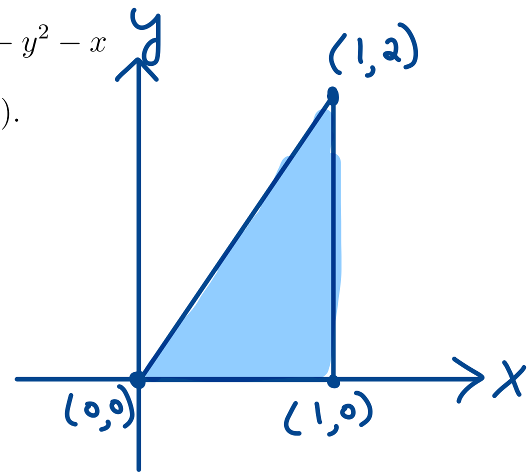
$f_{xy}(x, y) =$ $\sin(xy) + xy\cos(xy)$

$f_{yy}(x, y) =$ $\frac{-2y}{(1+y^2)^2} + x^2\cos(xy)$

4. *(13 points)* Find the absolute minimum and maximum values of the function

$$f(x,y) = x^3 + 2xy - y^2 - x$$

over the triangle with vertices $(0,0)$, $(1,0)$, and $(1,2)$.



Crit. points:

$f_x(x,y) = 3x^2 + 2y - 1 = 0$

$f_y(x,y) = 2x - 2y = 0 \rightarrow x = y$

$3x^2 + 2x - 1 = 0$

$(3x-1)(x+1) = 0$

$x = \frac{1}{3}$, $y = \frac{1}{3}$

$x = -1$ (out of domain)

Only one crit. pt!

Boundary

Bottom: $f(x,0) = x^3 - x$

$f'(x) = 3x^2 - 1 = 0$

$x = \sqrt{\frac{1}{3}}$

Right: $f(1,y) = 2y - y^2$

$f'(y) = 2 - 2y = 0$

$y = 1$

Upper-left: $f(x,2x) = x^3 + 4x^2 - 4x^2 - x$

$= x^3 - x$

$f'(x) = 3x^2 - 1 = 0$

$x = \frac{1}{\sqrt{3}}$, $y = \frac{2}{\sqrt{3}}$

Also check endpoints: $(0,0)$, $(1,0)$, $(1,2)$

Points to check

$f\left(\frac{1}{3}, \frac{1}{3}\right) = \frac{-5}{27} \approx -0.185$

$f\left(\frac{1}{\sqrt{3}}, 0\right) = \frac{-2}{3\sqrt{3}} \approx -0.385$

$f(1,1) = 1$

$f\left(\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right) = \frac{-2}{3\sqrt{3}}$

$f(0,0) = 0$

$f(1,0) = 0$

$f(1,2) = 0$

Absolute minimum: $\frac{-2}{3\sqrt{3}}$

Absolute maximum: 1

5. *(12 points)* Reverse the order of integration and evaluate the integral:

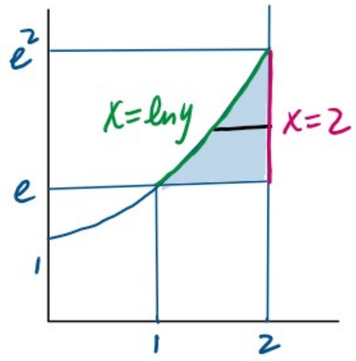
$$\int_e^{e^2}\int_{\ln y}^{2}(e^x - ex)^5\,dx\,dy$$

D: given as type II region:

$$\ln y \le x \le 2$$
$$e \le y \le e^2$$

left: $x = \ln y \iff y = e^x$

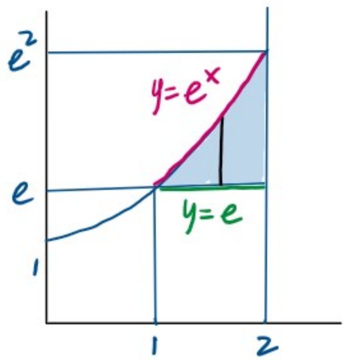
right: $x = 2$



D: expressed as type I:

$$e \le y \le e^x$$

bottom $1 \le x \le 2$ top



$$\int_1^2\int_e^{e^x}(e^x - ex)^5\,dy\,dx$$

$$=\int_1^2\left[(e^x-ex)^5 y\Big|_{y=e}^{y=e^x}\right]dx$$

$$=\int_1^2 (e^x-ex)^5(e^x-e)\,dx$$

$u = e^x - ex$

$du = e^x - e$

$\int u^5\,du = \frac{u^6}{6}$

$$= \frac{1}{6}(e^x-ex)^6\Big|_{x=1}^{x=2}$$

$$=\frac{1}{6}\left[(e^2-2e)^6-(e-e)^6\right]$$

$$=\frac{1}{6}(e^2-2e)^6$$

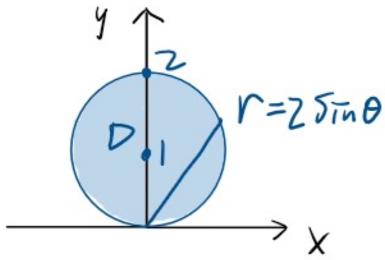
Answer: ____________________

6. (13 *points)* Find the volume of the solid that lies below the cone $z = 2 + \sqrt{x^2 + y^2}$, above the xy-plane, and is enclosed by the cylinder $x^2 + (y-1)^2 = 1$.

You may find the following the trig identities helpful:

$$\cos^2(t) = \frac{1+\cos(2t)}{2} \qquad \text{and} \qquad \sin^2(t) = \frac{1-\cos(2t)}{2}$$

$$V = \iint_D 2 + \sqrt{x^2+y^2}\, dA$$



D in Polar: $\begin{cases} 0 \le r \le 2\sin\theta \\ 0 \le \theta \le \pi \end{cases}$

Cone: $z = 2 + \sqrt{x^2+y^2} \rightsquigarrow z = 2 + r$

$$V = \int_0^{\pi}\int_0^{2\sin\theta} (2+r)\, r\, dr\, d\theta$$

$$= \int_0^{\pi} \left(r^2 + \frac{r^3}{3} \right)\Big|_{r=0}^{r=2\sin\theta} d\theta$$

$$= \int_0^{\pi} 4\sin^2\theta + \frac{8}{3}\sin^3\theta \, d\theta$$

$$= \int_0^{\pi} \frac{4}{2}(1-\cos(2\theta)) + \frac{8}{3}(1-\cos^2\theta)\sin\theta \, d\theta$$

$$= \quad 2\theta - 2\cdot\frac{1}{2}\sin(2\theta) + \frac{8}{3}\left[-\cos\theta + \frac{\cos^3\theta}{3}\right]\Big|_0^{\pi}$$

$$= 2\pi - (0-0) + \frac{8}{3}\left(-(-1-1) + \frac{1}{3}(-1-1)\right)$$

$$= 2\pi + \frac{8}{3}\left(2 - \frac{2}{3}\right)$$

$$= 2\pi + \frac{32}{9}$$

Volume: ____________________

7. For this problem, let $f(x) = \ln(2x-1)$.

   (a) *(5 points)* Find the third Taylor polynomial, $T_3(x)$, for the function f based at $b = 1$.

$f(x) = \ln(2x-1)$ $\quad f(1) = 0$

$f'(x) = \frac{2}{2x-1}$ $\quad f'(1) = 2$

$f''(x) = \frac{-4}{(2x-1)^2}$ $\quad f''(1) = -4$

$f'''(x) = \frac{16}{(2x-1)^3}$ $\quad f'''(1) = 16$

$T_3(x) =$ $\boxed{2(x-1) - 2(x-1)^2 + \frac{8}{3}(x-1)^3}$

   (b) *(4 points)* Use $T_3(x)$ to approximate the value of $\ln(1.04)$.

$\ln(2x-1) = \ln(1.04)$

$2x-1 = 1.04$

$x = 1.02$

$\ln(1.04) = f(1.02) \approx T_3(1.02)$

$= 2(0.02) - 2(0.02)^2 + \frac{8}{3}(0.02)^3$

$\ln(1.04) \approx$ $\boxed{0.039221}$

   (c) *(5 points)* Use Taylor's inequality to find an upper bound (as sharp as possible) for the error in your approximation in part (b).

$\left|f^{(4)}(x)\right| = \left|\frac{-96}{(2x-1)^4}\right| \leftarrow$ largest on $[1, 1.02]$ when x is smallest

(ok to use $[0.98, 1.02]$ instead)

can use $M = 96$

$|T_3(x) - f(x)| \le \frac{1}{24}(96)|1.02-1|^4 = 6.4 \times 10^{-7}$

Upper bound: $\boxed{0.00000064}$

8. For this problem, you may use the following basic Taylor series:

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k, \quad e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad \sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}, \quad \cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

For parts (a)–(c), let $f(x) = x \arctan(2x^2)$

(a) *(6 points)* Give the Taylor series for f based at $b = 0$ using one sigma sign.

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$$

$$\downarrow x \to -x^2$$

$$\frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-1)^k x^{2k}$$

$$\downarrow \int$$

$$\arctan(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}$$

$$x \to 2x^2$$

$$\arctan(2x^2) = \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k+1} x^{4k+2}}{2k+1}$$

$$x \arctan(2x^2) = \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k+1} x^{4k+3}}{2k+1}$$

Taylor series: $\boxed{\sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k+1} x^{4k+3}}{2k+1}}$

(b) *(3 points)* Find the largest open interval on which this Taylor series converges.

Original: $-1 < x < 1$

Sub $x \to -x^2$: $-1 < -x^2 < 1$

Sub $x \to 2x^2$: $-1 < -(2x^2)^2 < 1$

$-1 < -4x^4 < 1$

$\frac{-1}{4} < x^4 < \frac{1}{4}$ $\quad \frac{-\sqrt{2}}{2} < x < \frac{\sqrt{2}}{2}$

Interval: $\boxed{\left(\frac{-\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)}$

(c) *(5 points)* Find $T_8(x)$, the 8[th] Taylor polynomial for f based at $b = 0$, and use it to approximate $\int_0^1 f(x)\,dx$.

$$\sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k+1} x^{4k+3}}{2k+1} = \underbrace{2x^3 - \frac{8}{3}x^7}_{T_8(x)} + \frac{32}{5}x^{11} + \ldots$$

$$\int_0^1 \left(2x^3 - \frac{8}{3}x^7\right) dx = \left(\frac{1}{2}x^4 - \frac{1}{3}x^8\right)\Big]_0^1 = \frac{1}{2} - \frac{1}{3}$$

$T_8(x) = \boxed{2x^3 - \frac{8}{3}x^7}$

$\int_0^1 f(x)\,dx \approx \boxed{\frac{1}{6}}$