1. *(6 points per part)*

   (a) Find parametric equations for the line of intersection of the planes $2x - y + 3z + 4 = 0$ and $-x + y - z = 0$.

+2 COMBINE OR SOLVE

① $2x - y + 3z + 4 = 0$

② $-x + y - z = 0 \rightarrow y = x + z$

COMBINE (ADD): $x + 2z = -4$

+2 FIND 2 PTS OR CROSS PRODUCT

ONE POINT: $P(0, -2, -2)$

ANOTHER PT: $Q(-4, -4, 0)$

+2 CORRECT FORM & CONCLUSION

$x = 0 - 4u$

$y = -2 - 2u$

$z = -2 + 2u$

$\uparrow$ P $\quad \uparrow$ $\overrightarrow{PQ}$

OR

ALSO CORRECT

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 3 \\ -1 & 1 & -1 \end{vmatrix}$$

$= (1-3)\vec{i} - (-2 - -3)\vec{j} + (2-1)k$

$= \langle -2, -1, 1 \rangle$

* DIRECTION: ANY VECTOR PARALLEL TO $\langle -2, -1, 1 \rangle$
* POINT: ANY POINT ON INTERSECTION OF PLANES

Parametric equations: $x = -2t,\ y = -2 - t,\ z = -2 + t$

   (b) Find the intersection point of the line you found in part (a) with the plane $x + 2y + z = 12$.

+2 SUBSTITUTE

$-2t + 2(-2 - t) + (-2 + t) = 12$

+2 SOLVE

$-2t - 4 - 2t - 2 + t = 12$

$-6 - 3t = 12$

$-3t = 18$

$t = -6$ ← WILL DEPEND ON (a)

+2 GIVE POINT

SAME FOREVEYONE

Intersection point: $(x, y, z) = (12, 4, -8)$

2. *(4 points per part)* Consider the curve given by the position function $\mathbf{r}_1(t) = \langle \ln(t), t^2+5, 3t \rangle$ for $t > 0$.

(a) Find the curvature at $t = 1$.

DERIV +1

$\vec{r}_1\,'(t) = \langle \frac{1}{t}, 2t, 3 \rangle \rightarrow \vec{r}_1\,'(1) = \langle 1, 2, 3 \rangle$

$\vec{r}_1\,''(t) = \langle -\frac{1}{t^2}, 2, 0 \rangle \rightarrow \vec{r}_1\,''(1) = \langle -1, 2, 0 \rangle$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ -1 & 2 & 0 \end{vmatrix}$$

CROSS +2 & length

$\vec{r}_1\,'(1) \times \vec{r}_1\,''(1) = (0-6)\vec{i} - (0 - -3)\vec{j} + (2 - -2)\vec{k}$
$= \langle -6, -3, 4 \rangle$

+1 FINAL FORMULA

$$\kappa(1) = \frac{\sqrt{36+9+16}}{\left(\sqrt{1+4+9}\right)^3} = \frac{\sqrt{61}}{\underbrace{\left(\sqrt{14}\right)^3}} \approx 0.149098$$

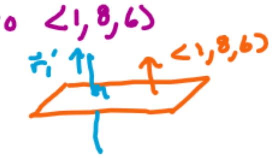
ASIDE: SAME AS $\frac{1}{14}\sqrt{\frac{61}{14}} = \sqrt{\frac{61}{2744}}$

$\kappa(1) =$ $\frac{\sqrt{61}}{14^{3/2}}$

(b) Find the (x, y, z) point on the curve at which the tangent line is orthogonal to the plane $x + 8y + 6z = 7$.

DERIV +1

WANT $\vec{r}_1\,'(t) = \langle \frac{1}{t}, 2t, 3 \rangle$ TO BE PARALLEL TO $\langle 1, 8, 6 \rangle$



$\langle \frac{1}{t}, 2t, 3 \rangle = k \langle 1, 8, 6 \rangle$

SOLVE +2

$\Rightarrow \frac{1}{t} = k$

$2t = 8k$ $\quad 2t = 4 \rightarrow t = 2$

$3 = 6k \rightarrow k = \frac{1}{2}$

POINT +1

WANT POINT ON CURVE WHEN $t = 2$

$\Rightarrow \vec{r}_1\,'(2) = \langle \ln(2), (2)^2 + 5, 3(2) \rangle$

$(x, y, z) =$ $(\ln(2), 9, 6)$

(c) Find the **acute** angle of intersection of $\mathbf{r}_1(t)$ with the curve $\mathbf{r}_2(u) = \langle 0, u^3 - u, \sqrt{u^2+5} \rangle$.

(Give your answer rounded to the nearest degree.)

+2 FINDING $u = 2$

INTERSECTION: ① $\ln(t) = 0 \longrightarrow t = 1$

② $t^2 + 5 = u^3 - u \rightarrow 6 = u^3 - u$

③ $3t = \sqrt{u^2+5} \rightarrow 3 = \sqrt{u^2+5}$

$\Rightarrow 9 = u^2 + 5$

$4 = u^2 \rightarrow u = \cancel{-2}$ NO

$\rightarrow u = 2$ YES

+2 ANGLE BETWEEN DERIVATIVES

$\vec{r}_1\,'(1) = \langle 1, 2, 3 \rangle$

$\vec{r}_2\,'(u) = \langle 0, 3u^2 - 1, \frac{2u}{2\sqrt{u^2+5}} \rangle$

$\vec{r}_2\,'(2) = \langle 0, 11, \frac{2}{3} \rangle$ $\quad \theta = \cos^{-1}\left(\frac{22+2}{\sqrt{14}\sqrt{121 + 4/9}} \right) \approx 54.4053$

Angle $\approx$ 54 degrees

m126finalWin2024 Q3 Q4 solution

3. *(6 points per part)* Consider the function $f(x, y) = x^2 + xy^2 - y$.

(a) Find all the saddle point(s) of f in $\mathbb{R}^2$. Justify your answer.

$f_x = 2x + y^2 = 0 \Rightarrow 2x = -y^2$

$f_y = 2xy - 1 = 0 \Rightarrow -y^3 - 1 = 0 \Rightarrow y = -1$, $x = -\frac{1}{2}$ cr. pt $(-\frac{1}{2}, -1)$

$f_{xx} = 2$

$f_{yy} = 2x$

$f_{xy} = 2y$

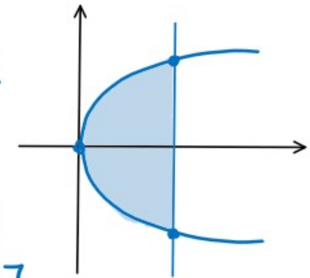
$D(x,y) = f_{xx}f_{yy} - f_{xy}^2 = 4x - (2y)^2$

$D(-\frac{1}{2}, -1) = -2 - 4 < 0 \Rightarrow$ saddle pt

Saddle point(s): $(-\frac{1}{2}, -1)$

(b) Let D be the closed region bounded by $x = y^2$ and $x = 1$. Find the absolute maximum and absolute minimum values of the function f on the region D, and the points where these extrema occur.

From (a), f has cr pt @ $(-\frac{1}{2}, -1)$, which is not in D.



On the boundary of D:

- parabola $x = y^2$: $f(x,y) = f(y^2, y) = y^4 + y^4 - y$

$g_1(y) = 2y^4 - y$, $y \in [-1, 1]$

$g_1'(y) = 8y^3 - 1 = 0 \Rightarrow y = \frac{1}{2}$,

$f(\frac{1}{4}, \frac{1}{2}) = g_1(\frac{1}{2}) = \frac{1}{8} - \frac{1}{2} = -\frac{3}{8}$

$f(1, 1) = g_1(1) = 2 - 1 = 1$

$f(1, -1) = g_1(-1) = 2 + 1 = 3$

- line $x = 1$ $f(x,y) = f(1, y) = 1 + y^2 - y$

$g_2(y) = 1 + y^2 - y$ $y \in [-1, 1]$ (end pts checked)

$g_2'(y) = 2y - 1 = 0 \Rightarrow y = \frac{1}{2}$,

$f(1, \frac{1}{2}) = g_2(\frac{1}{2}) = \frac{3}{4}$

Absolute (global) maximum on D: $f(1, -1) = 3$

Absolute (global) minimum on D: $f(\frac{1}{4}, \frac{1}{2}) = -\frac{3}{8}$

4. (*12 points*) Consider the surface $x^2 - (y+1)^2 - 3z^2 = 2$.

   Find the point(s) on the surface that are closest to the point $A(0,3,1)$.

   For full credit, you must show work to justify your answer.

minimize: $d = \sqrt{(x-0)^2 + (y-3)^2 + (z-1)^2}$

constraint: $x^2 - (y+1)^2 - 3z^2 = 2$

$\Rightarrow x^2 = 2 + (y+1)^2 + 3z^2$

substitute

$f(x,y,z) = d^2 = x^2 + (y-3)^2 + (z-1)^2$

$g(y,z) = 2 + (y+1)^2 + 3z^2 + (y-3)^2 + (z-1)^2$

since $(y,z) \in \mathbb{R}^2$ on the surface, domain of $g(y,z)$ has no boundary.

The abs. min must occur at critical pt

$g_y = 2(y+1) + 2(y-3) = 4y - 4 = 0 \Rightarrow y = 1$

$g_z = 6z + 2(z-1) = 8z - 2 = 0 \Rightarrow z = \frac{1}{4}$

$x^2 = 2 + (y+1)^2 + 3z^2$

$= 2 + 4 + \frac{3}{16} = \frac{99}{16}$

$\Rightarrow x = \pm \frac{\sqrt{99}}{4}$

Closest point(s): $\left(\pm\frac{\sqrt{99}}{4}, 1, \frac{1}{4}\right)$

5. *(6 points per part)* The parts of this question are not related.

(a) The region D is bounded above by the curve $y = \frac{1}{x}$, below by the line $y = \frac{1}{2}(x-1)$ and to left by the line $x = 1$.

Set up the integral $\iint_D f(x,y)\,dA$ using **two different orders of integration**.

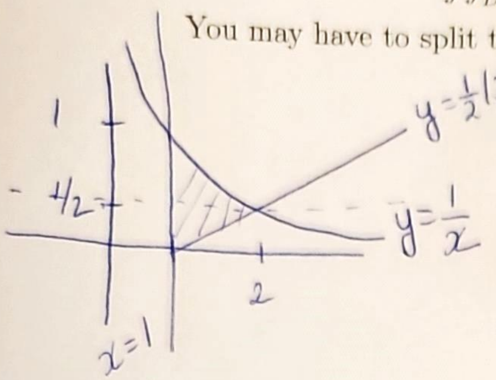
You may have to split the region.

$\frac{1}{2}(x-1) = \frac{1}{x} \rightarrow x^2 - x = 2$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x = 2,\ x = -1$



One way: $\int_1^2 \int_{\frac{1}{2}(x-1)}^{1/x} f(x,y)\,dy\,dx$

Another way: $\int_0^{1/2} \int_1^{2y+1} f(x,y)\,dx\,dy + \int_{1/2}^{1} \int_1^{1/y} f(x,y)\,dx\,dy$

(b) Evaluate $\int_0^1 \int_{y^4}^{\sqrt{y}} (\sqrt{x} - y^2)\,dx\,dy$.

$= \int_0^1 \frac{2}{3}x^{3/2} - xy^2 \Big|_{y^4}^{\sqrt{y}} dy = \int_0^1 \left(\frac{2}{3}y^{3/4} - y^{5/2}\right) - \left(\frac{2}{3}y^6 - y^6\right) dy$

$= \frac{2}{3}\frac{4}{7}y^{7/4} - \frac{2}{7}y^{7/2} + \frac{1}{3}\frac{1}{7}y^7 \Big|_0^1 = \frac{8}{21} - \frac{2}{7} + \frac{1}{21} = \frac{8-6+1}{21} = \frac{3}{21} = \frac{1}{7}$

Answer: $1/7$

6. *(12 points)* Find the volume of the region that is inside the ellipsoid $x^2 + y^2 + 4z^2 = 25$ and above the cone $z = \sqrt{x^2 + y^2}$.

$$\left(\sqrt{x^2+y^2}\right)^2 = \frac{-x^2 - y^2 + 25}{4}$$

$$4x^2+4y^2+x^2+y^2 = 25$$

$$x^2+y^2 = 5 = (\sqrt{5})^2$$

$$\iint_{x^2+y^2\le 5} \frac{\sqrt{25-x^2-y^2}}{2} - \sqrt{x^2+y^2}\, dA$$

$$= \int_0^{2\pi}\int_0^{\sqrt{5}} \left(\frac{\sqrt{25-r^2}}{2} - r\right) r\, dr\, d\theta$$

$$= 2\pi \left[-\frac{1}{6}(25-r^2)^{3/2} - \frac{r^3}{3} \Big|_0^{\sqrt{5}} \right]$$

$$= 2\pi \left[-\frac{1}{6}(25-5)^{3/2} - \frac{5\sqrt{5}}{3} + \frac{1}{6}\cdot 25^{3/2} + 0 \right]$$

$$= 2\pi \left[-\frac{1}{6}\cdot 8\cdot 5\sqrt{5} - \frac{5\sqrt{5}}{3} + \frac{125}{6} \right] = 2\pi \left[-\frac{25\sqrt{5}}{3} + \frac{125}{6} \right]$$

$$= \frac{\pi}{3}\left[125 - 50\sqrt{5}\right]$$

$$= \frac{25\pi}{3}\left[5 - 2\sqrt{5}\right]$$

Volume = ______________________

7. Let $f(x) = \sqrt{(2x+1)^3}$.

(a) *(6 points)* Find the second Taylor polynomial $T_2(x)$ for f based at $b = 4$.

$f(x) = (2x+1)^{3/2}$ $\quad f(4) = 27$

$f'(x) = 3(2x+1)^{1/2}$ $\quad f'(4) = 9$

$f''(x) = 3(2x+1)^{-1/2}$ $\quad f''(4) = 1$

$T_2(x) = \boxed{27 + 9(x-4) + \frac{1}{2}(x-4)^2}$

(b) *(3 points)* Use your answer to part (a) to approximate $\sqrt{9.1^3}$.

$\sqrt{9.1^3} = \sqrt{(2(4.05)+1)^3} \approx T_2(4.05) = 27 + 9(.05) + \frac{1}{2}(.05)^2$

$9.1 = 2x+1$

$\downarrow$

$x = 4.05$

$\sqrt{9.1^3} \approx \boxed{27.45125}$

(c) *(5 points)* Use Taylor's inequality to find an upper bound (as sharp as possible) for the error in your approximation in part (b).

$|f'''(x)| = \left|-3(2x+1)^{-3/2}\right| = \dfrac{3}{\sqrt{(2x+1)^3}}$ max when denom. is small $(x=4)$

On interval $[4, 4.05]$, use $M = \dfrac{3}{\sqrt{9^3}} = \dfrac{1}{9}$

$\left|T_2(x) - f(x)\right| \leq \frac{1}{6} M |x-4|^3 = \frac{1}{6}\,\frac{1}{9}\,(.05)^3 = \frac{1}{432000}$

Upper bound: $\boxed{\dfrac{1}{432000}}$

8. For this problem, let $f(x) = x\sin(x^4) + \dfrac{x^2}{4-x^3}$.

(a) *(6 points)* Find $T_{10}(x)$, the 10th Taylor polynomial for f based at $b = 0$.

$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$

$\downarrow x \to x^4$

$\sin(x^4) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{8k+4}}{(2k+1)!}$

$\downarrow \cdot x$

$x\sin(x^4) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{8k+5}}{(2k+1)!}$

$= x^5 - \frac{x^{13}}{6} + \dots$

$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$

$\downarrow \div 4$

$\frac{1}{4-4x} = \sum_{k=0}^{\infty} \frac{x^k}{4}$

$\downarrow x \to \frac{x^3}{4}$

$\frac{1}{4-x^3} = \sum_{k=0}^{\infty} \frac{x^{3k}}{4^{k+1}}$

$\cdot x^2$

$\sum_{k=0}^{\infty} \frac{x^{3k+2}}{4^{k+1}}$

$= \frac{x^2}{4} + \frac{x^5}{16} + \frac{x^8}{64} + \frac{x^{11}}{256} + \dots$

$T_{10}(x) =$ sum of terms w/ degree ≤ 10

$T_{10}(x) = \boxed{\frac{x^2}{4} + \frac{17}{16}x^5 + \frac{x^8}{64}}$

(b) *(3 points)* Find the largest open interval on which the Taylor series for f based at $b = 0$ converges.

$\sin x$ converges everywhere, & no changes from our operations.

$\frac{1}{1-x}$ converges for $-1 < x < 1$

$\downarrow x \to \frac{x^3}{4}$

$-1 < \frac{x^3}{4} < 1$

$-4 < x^3 < 4$

$-\sqrt[3]{4} < x < \sqrt[3]{4}$

Interval: $\boxed{(-\sqrt[3]{4}, \sqrt[3]{4})}$

(c) *(5 points)* Find $f^{2024}(0)$. (That is, the 2024th derivative of f at 0.) Give an exact answer.

Look for x^{2024} terms in T. series.

$\sum_{k=0}^{\infty} \frac{(-1)^k x^{8k+5}}{(2k+1)!} \to 8k+5 = 2024 \to 8k = 2019 \to$ no x^{2024} term (no int. k)

$\sum_{k=0}^{\infty} \frac{x^{3k+2}}{4^{k+1}} \to 3k+2 = 2024 \to 3k = 2022 \to k = 674$

$\frac{x^{2024}}{4^{675}} = \frac{f^{(2024)}(0)x^{2024}}{2024!}$

$f^{(2024)}(0) = \boxed{\frac{2024!}{4^{675}}}$