1. (*4 points per part*) For parts (a)–(c), consider the plane $\mathcal{P}: 2x + y + 3z = 2$ and the line $L:\ x = 1 - 4t,\ y = 2t,\ z = 5 + t$.

   (a) Find the acute angle between the line L and the line $y = 2, z = 6$.

   The line L has a direction vector $\mathbf{v} = \langle -4, 2, 1\rangle$.
   The line $y = 2, z = 6$ is parallel to the x-axis, it has a direction vector $\mathbf{i} = \langle 1, 0, 0\rangle$.

$$\cos^{-1}\left(\frac{\mathbf{v}\cdot\mathbf{i}}{|\mathbf{v}||\mathbf{i}|}\right) = \frac{-4}{\sqrt{21}}$$

   The acute angle is $\theta = \pi - \cos^{-1}\left(\frac{-4}{\sqrt{21}}\right)$ or $\cos^{-1}\left(\frac{4}{\sqrt{21}}\right)$.

   (b) Find the equation of the plane that contains the line L and is perpendicular to the plane $\mathcal{P}$.

   Since the unknown plane contains the line L, it contains any point on L, for example, at $t = 0$, $P_0(1, 0, 5)$.

   The normal vector of the unknown plane is perpendicular to both the line vector $\mathbf{v} = \langle -4, 2, 1\rangle$ and the normal vector of the given plane $\mathbf{n} = \langle 2, 1, 3\rangle$.

   Therefore a normal vector of the unknown plane is

$$\mathbf{v}\times\mathbf{n} = \langle -4, 2, 1\rangle \times \langle 2, 1, 3\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 2 & 1 \\ 2 & 1 & 3 \end{vmatrix} = \langle 5, 14, -8\rangle$$

   The plane equation is $5(x-1) + 14(y-0) - 8(z-5) = 0$. (answer not unique)

   (c) Find two points A and B on the plane $\mathcal{P}$ such that the vector $\overrightarrow{AB}$ is perpendicular to the line L.

   A vector $\overrightarrow{AB}$ lying on (parallel to) the plane $\mathcal{P}$ is perpendicular to the normal vector of the plane, since $\overrightarrow{AB}$ is also perpendicular to the line L, we may find

$$\overrightarrow{AB} = \mathbf{v}\times\mathbf{n} = \langle -4, 2, 1\rangle \times \langle 2, 1, 3\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 2 & 1 \\ 2 & 1 & 3 \end{vmatrix} = \langle 5, 14, -8\rangle \ \text{(already found in (b))}$$

   Then it suffices to find any point A on the plane $\mathcal{P}$ and then find the point B such that $\overrightarrow{AB} = \langle 5, 14, -8\rangle$. For example,

   $A(1, 0, 0)$ and $B(6, 14, -8)$ (answer not unique)

2. (*4 points per part*) For parts (a)–(c), a particle is moving along a space curve with initial position vector $\mathbf{r}(0) = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$. The velocity vector of the particle is

$$\mathbf{v}(t) = 3\cos(t)\,\mathbf{i} + 4t\,\mathbf{j} + \sin(t)\,\mathbf{k}$$

(a) Find the position vector function $\mathbf{r}(t)$.

$\mathbf{r}(t) = \displaystyle\int \mathbf{v}(t)\,dt = \langle 3\sin(t) + C_1, 2t^2 + C_2, -\cos(t) + C_3\rangle.$

Since $\mathbf{r}(0) = \langle C_1, C_2, -1 + C_3\rangle = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$, $C_1 = 2, C_2 = 1, C_3 = 0,$

$\mathbf{r}(t) = \langle 3\sin(t) + 2, 2t^2 + 1, -\cos(t)\rangle.$

(b) Find the equation of a surface that contains the space curve $\mathbf{r}(t)$.

Since $\sin(t) = \dfrac{x-2}{3}, \cos(t) = -z,$

$\mathbf{r}(t)$ is contained in the cylinder $\left(\dfrac{x-2}{3}\right)^2 + z^2 = 1.$

(c) Find the curvature of the space curve $\mathbf{r}(t)$ at $t = \pi$.

$\mathbf{r}'(\pi) = \mathbf{v}(\pi) = \langle -3, 4\pi, 0\rangle,$

$\mathbf{r}''(t) = \mathbf{v}'(t) = \langle -3\sin(t), 4, \cos(t)\rangle$, $\mathbf{r}''(\pi) = \langle 0, 4, -1\rangle,$

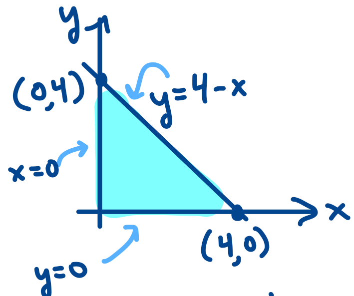
$$\mathbf{r}'(\pi) \times \mathbf{r}''(\pi) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 4\pi & 0 \\ 0 & 4 & -1 \end{vmatrix} = \langle -4\pi, -3, -12\rangle,$$

$$\kappa(\pi) = \frac{|\mathbf{r}'(\pi) \times \mathbf{r}''(\pi)|}{|\mathbf{r}'(\pi)|^3} = \frac{|\langle -4\pi, -3, -12\rangle|}{|\langle -3, 4\pi, 0\rangle|^3} = \frac{\sqrt{153 + 16\pi^2}}{(9 + 16\pi^2)^{3/2}}.$$

3. (*12 points*) Let f be the function given by:

$$f(x, y) = x^2 - y^2 - 2xy + 4y,$$

and let A, B, and C be the points $(0, 0)$, $(0, 4)$ and $(4, 0)$. Find the global minimum and maximum of f in the filled triangle ABC.



Critical points: $f_x(x,y) = 2x - 2y = 0 \to x = y$

$f_y(x,y) = -2y - 2x + 4 = 0 \to 4x = 4 \to x = y = 1$

$(1,1)$ is a crit. point.

Left edge: $x=0$

$f(0,y) = -y^2 + 4y$

$f'(y) = -2y + 4 = 0 \to y = 2$ Check $(0,2)$

Bottom edge: $y=0$

$f(x,0) = x^2$

$f'(x) = 2x = 0 \to x = 0$ Check $(0,0)$

Top-right edge: $y = 4-x$

$f(x, 4-x) = x^2 - (4-x)^2 - 2x(4-x) + 4(4-x) = 2x^2 - 4x$

$f'(x) = 4x - 4 = 0 \to x = 1$, Check $(1,3)$.

Also check endpoints $(0,0)$, $(0,4)$, $(4,0)$.

Plug in:

$f(0,0) = 0$

$f(1,1) = 2$

$f(0,4) = 0$

$f(4,0) = \boxed{16} \leftarrow$ max

$f(0,2) = 4$

$f(1,3) = \boxed{-2} \leftarrow$ min

4. (*12 points*) Find a point on the surface of equation $z = x^2 - y^2$ whose tangent plane is perpendicular to the line of equation

$$x = t$$
$$y = 3t$$
$$z = -2t.$$

direction $\langle 1, 3, -2 \rangle$

$\frac{\partial z}{\partial x} = 2x$

$\frac{\partial z}{\partial y} = -2y$

Normal vector $\langle 2x, -2y, -1 \rangle$

So $\langle 1, 3, -2 \rangle$ is parallel to $\langle 2x, -2y, -1 \rangle$

$\langle 1, 3, -2 \rangle = c \langle 2x, -2y, -1 \rangle$

$c = 2$

$\langle 1, 3, -2 \rangle = \langle 4x, -4y, -2 \rangle$

$x = \frac{1}{4}$, $y = \frac{-3}{4}$, $z = x^2 - y^2 = \frac{-1}{2}$

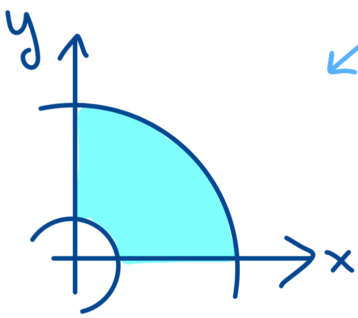
So the point of tangency is $\boxed{\left(\frac{1}{4}, \frac{-3}{4}, \frac{-1}{2}\right)}$

5. (*12 points*) Find the volume of the solid below the surface

$$z = \frac{2x + y + 1}{\sqrt{x^2 + y^2}}$$

and above the region in the xy-plane

$$D = \{(x, y) \mid x \geq 0, y \geq 0, 1 \leq x^2 + y^2 \leq 4\}.$$



Hm, looks polar.

$$\int_0^{\frac{\pi}{2}} \int_1^2 \frac{2r\cos\theta + r\sin\theta + 1}{\cancel{r}} \cancel{r}\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_1^2 \left(2r\cos\theta + r\sin\theta + 1\right) dr\, d\theta = \int_0^{\frac{\pi}{2}} \left(r^2\cos\theta + \frac{1}{2}r^2\sin\theta + r\right)\Big]_{r=1}^{r=2} d\theta$$

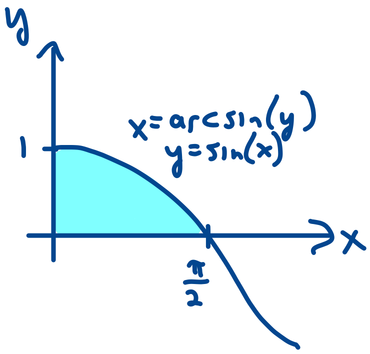
$$= \int_0^{\frac{\pi}{2}} \left(3\cos\theta + \frac{3}{2}\sin\theta + 1\right) d\theta = \left(3\sin\theta - \frac{3}{2}\cos\theta + \theta\right)\Big]_0^{\frac{\pi}{2}}$$

$$= \left(3 + \frac{\pi}{2}\right) - \left(\frac{-3}{2}\right)$$

$$= \boxed{\frac{9 + \pi}{2}}$$

6. (*12 points*) Evaluate the integral

$$\int_0^1 \int_{\arcsin y}^{\pi/2} \cos x\sqrt{1+\cos^2 x}\,dx\,dy.$$



Impossible.

Must reverse order of integration.

$$\int_0^{\frac{\pi}{2}} \int_0^{\sin(x)} \cos(x)\sqrt{1+\cos^2 x}\,dy\,dx$$

$$= \int_0^{\frac{\pi}{2}} \left(y \cos x \sqrt{1+\cos^2 x} \right)\Bigg]_{y=0}^{y=\sin x} dx$$

$$= \int_0^{\pi/2} \sin x \cos x \sqrt{1+\cos^2 x}\,dx$$

$u = 1+\cos^2 x$

$du = -2\sin x \cos x\,dx$

$$= \int_2^1 \frac{-1}{2}\sqrt{u}\,du = \frac{-1}{3} u^{3/2}\Big]_2^1 = \frac{-1}{3}\left(1-2\sqrt{2}\right)$$

$$= \boxed{\frac{2\sqrt{2}-1}{3}}$$

7. For parts (a)–(c), let $f(x) = \int_2^x \cos(\pi t^2)\, dt$.

(a) *(5 points)* Find the second Taylor polynomial, $T_2(x)$, for $f(x)$ based at $b = 2$.

$f(x) = \int_2^x \cos(\pi t^2)\,dt \qquad f(2) = 0$

$f'(x) = \cos(\pi x^2) \qquad f'(2) = 1$

$f''(x) = -2\pi x \sin(\pi x^2) \qquad f''(2) = 0$

$\boxed{T_2(x) = (x-2)}$

(b) *(3 points)* Use your answer to part (a) to approximate $\int_2^{2.1} \cos(\pi t^2)\, dt$.

$\int_2^{2.1} \cos(\pi t^2)\,dt = f(2.1) \approx T_2(2.1) = 2.1 - 2 = \boxed{0.1}$

(c) *(6 points)* Find an upper bound on the error for your answer from part (b).

(Note: For full credit, you do not need to find a tight upper bound, but you must justify your answer with Taylor's inequality.)

$f'''(x) = -2\pi \sin(\pi x^2) - 4\pi^2 x^2 \cos(\pi x^2)$

$|f'''(x)| \le \underbrace{|2\pi \sin(\pi x^2)|}_{\le 2\pi} + \underbrace{|4\pi^2 x^2 \cos(\pi x^2)|}_{\le 17.64\pi^2 \text{ on } [2,2.1]} \le 2\pi + 17.64\pi^2$

(other answers are possible but note that neither $|f'''(2)|$ nor $|f'''(2.1)|$ is an upper bound)

So $|T_2(x) - f(x)| \le \boxed{\frac{1}{6}(2\pi + 17.64\pi^2)(0.1)^3} \approx 0.03006$

8. For this problem, you may use the following basic Taylor series:

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k, \quad e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad \sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}, \quad \cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

(a) *(6 points)* Find the Taylor series for $f(x) = \frac{x^3}{1+x^4} - 3x\sin(x^2)$ based at $b = 0$. Express your answer using $\sum$–notation.

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$$

$x \to -x^4$

$$\frac{1}{1+x^4} = \sum_{k=0}^{\infty} (-1)^k x^{4k}$$

$$\frac{x^3}{1+x^4} = \sum_{k=0}^{\infty} (-1)^k x^{4k+3}$$

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

$x \to x^2$

$$\sin(x^2) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+2}}{(2k+1)!}$$

$$-3x\sin(x^2) = \sum_{k=0}^{\infty} \frac{-3(-1)^k x^{4k+3}}{(2k+1)!}$$

$$\frac{x^3}{1+x^4} - 3x\sin(x^2) = \boxed{\sum_{k=0}^{\infty} (-1)^k \left(1 - \frac{3}{(2k+1)!}\right) x^{4k+3}}$$

(b) *(3 points)* Find the open interval of convergence for the series you found in (a).

$\frac{1}{1-x}$ converges for $-1 < x < 1$

$x \to -x^4$

$-1 < -x^4 < 1$

sin(x) converges for all x

still $\boxed{(-1,1)}$

(c) *(5 points)* Find $f^{(2023)}(0)$, i.e. the 2023rd derivative of f at 0.

x^{2023} term of occurs at $4k+3 = 2023 \to k = 505$

$$\frac{f^{(2023)}(0)}{2023!} x^{2023} = (-1)^{505}\left(1 - \frac{3}{1011!}\right) x^{2023}$$

$$f^{(2023)}(0) = \boxed{-\left(1 - \frac{3}{1011!}\right)(2023!)}$$