1. *(2 points each)* Each of the following multiple choice problems has one correct answer. Circle it. You do not need to show any reasoning.

   (a) What is the radius of the sphere through $(-1, 2, 0)$ with center $(0, 0, 2)$?

   (i) $\sqrt{2}$ (ii) 3 (iii) 4 (iv) $\sqrt{5}$

   radius = distance = $\sqrt{1^2+2^2+2^2} = 3$

   (b) If $\mathbf{proj_a b} = \langle 1, 2, 3\rangle$, what's $\mathbf{proj}_{\langle 2,4,6\rangle}\mathbf{b}$?

   (i) $\langle 1, 2, 3\rangle$ (ii) $\langle 2, 4, 6\rangle$ (iii) $\langle 3, 6, 9\rangle$ (iv) $\langle \sqrt{2}, 2\sqrt{2}, 3\sqrt{2}\rangle$

   $\vec{a}$ and $\langle 1,2,3\rangle$ are parallel, so these are the same!

   (c) If **a** and **b** are unit vectors, which of the following <u>**cannot**</u> equal $\mathbf{a} \times \mathbf{b}$?

   (i) $\langle 0, 0, 0\rangle$ (ii) $\langle 0, 1, 0\rangle$ (iii) $\langle 1, 0, 1\rangle$ (iv) $\langle \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\rangle$

   $|\vec{a}\times\vec{b}| = \underbrace{|\vec{a}|}_{1}\underbrace{|\vec{b}|}_{1}\sin\theta$, so the magnitude must be ≤ 1.

   magnitude $= \sqrt{2}$

   (d) Which point lies on the line through $(1, 2, 3)$ and $(2, 0, 1)$?

   (i) $(-1, 5, 7)$ (ii) $(0, 4, 5)$ (iii) $(2, 1, 2)$ (iv) $(3, -2, 5)$

   $(1,2,3)$ is half way between this and $(2,0,1)$

   (e) Which of the following planes is parallel to the line $\mathbf{r}(t) = \langle t, 2t, 3t\rangle$?

   (i) $x + y + z = 6$ (ii) $x + 2y + 3z = 1$ (iii) $2x - y + z = 4$ (iv) $x + y - z = 7$

   Normal vector should be <u>orthogonal</u> to $\langle 1,2,3\rangle$

   $\langle 1,2,3\rangle \cdot \langle 1,1,-1\rangle = 0$

2. *(6 points per part)* **Parts (a) and (b) are unrelated.**

   (a) Find an equation for the surface consisting of all points (x, y, z) such that the distance from (x, y, z) to the x-axis is equal to the distance from (x, y, z) to the point $(0, 0, 2)$, **and** give a precise name for the corresponding 3D surface.

$$\sqrt{y^2+z^2} = \sqrt{x^2+y^2+(z-2)^2}$$

$$\cancel{y^2} + \cancel{z^2} = x^2 + \cancel{y^2} + \cancel{z^2} - 4z + 4$$

$$4z = x^2 + 4$$

Equation: $\boxed{4z = x^2 + 4}$

Name: $\boxed{\text{Parabolic cylinder}}$

   (b) Find a vector function $\mathbf{r}(t)$ whose space curve is the curve of intersection of the cylinder $y^2 + z^2 = 1$ and the hyperbolic paraboloid $x - y^2 + z^2 = 2$.

$y = \cos t$

$z = \sin t$

$x = 2 + y^2 - z^2$

$x = 2 + \cos^2 t - \sin^2 t$

$\mathbf{r}(t) = \boxed{\left\langle 2 + \cos^2 t - \sin^2 t, \cos t, \sin t \right\rangle}$

3. *(6 points per part)* For parts (a) and (b), let $\mathcal{S}$ be the surface $z = \dfrac{y}{x} + \sqrt{y - 2x}$.

(a) Find the equation of the plane tangent to $\mathcal{S}$ at $(2, 8, 6)$.

$$\frac{\partial z}{\partial x} = \frac{-y}{x^2} - \frac{1}{\sqrt{y-2x}}$$

$$\frac{\partial z}{\partial y} = \frac{1}{x} + \frac{1}{2\sqrt{y-2x}}$$

$x=2$
$y=8$

$$\frac{\partial z}{\partial x} = \frac{-5}{2}$$

$$\frac{\partial z}{\partial y} = \frac{3}{4}$$

Plane equation: $\boxed{z = \frac{-5}{2}(x-2) + \frac{3}{4}(y-8) + 6}$

(b) Use linearization to estimate the x-coordinate of the point $(x, 7.98, 5.91)$ on $\mathcal{S}$.

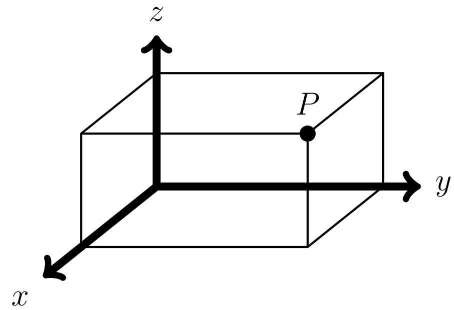
$$5.91 = \frac{-5}{2}(x-2) + \frac{3}{4}(7.98-8) + 6$$

$$\frac{-5}{2}(x-2) = -0.075$$

$x \approx$ $\boxed{x = 2.03}$

4. *(13 points)* Find the volume of the largest rectangular box in the first octant with three faces in the coordinate planes and the vertex opposite the origin (i.e. the one marked P in the picture below) on the surface $2x^2 + y + z = 90$.

In order to receive full credit, you must show that your answer really is the maximum.



Let $P = (x, y, z)$, so volume is

$V = xyz$ with $2x^2 + y + z = 90$

$z = 90 - 2x^2 - y$

Want max of

$f(x,y) = xy(90 - 2x^2 - y)$

$= 90xy - 2x^3y - xy^2$

$f_x(x,y) = 90y - 6x^2y - y^2 = 0$

$f_y(x,y) = 90x - 2x^3 - 2xy = 0$

ok to divide by x or y, b/c x or y will never be zero @ a max volume

$90 - 6x^2 - y = 0 \rightarrow y = 90 - 6x^2$

$90 - 2x^2 - 2y = 0 \rightarrow 90 - 2x^2 - 2(90 - 6x^2) = 0$

$10x^2 = 90$

$x^2 = 9$

$x = 3$ positive b/c first octant

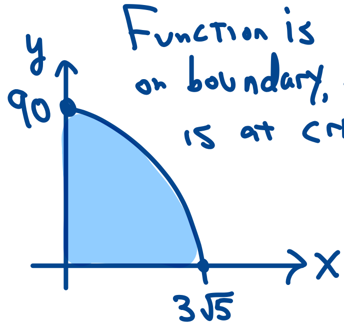
$y = 90 - 6(3)^2 = 36$

$z = 90 - 2(3)^2 - 36 = 36$

So $x = 3$, $y = 36$, $z = 36$

This is the max because the domain $x \geq 0$, $y \geq 0$, $90 - 2x^2 - y \geq 0$ is closed & bounded.

Function is 0 on boundary, so max is at crit point!



Maximum volume: $\boxed{3888}$

5. *(12 points)* Set up, but **do not evaluate**, an iterated integral using polar coordinates equal to the volume between the paraboloids $z = -2x + 3x^2 + 3y^2$ and $z = 2x^2 + 2y^2$.

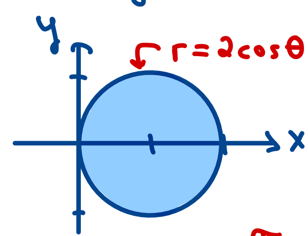
Intersection is

$-2x + 3x^2 + 3y^2 = 2x^2 + 2y^2$

$x^2 - 2x + 1 + y^2 = 0 + 1$

$(x-1)^2 + y^2 = 1$ $\rightarrow$

Region of integration:



$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$0 \le r \le 2\cos\theta$

In this region, $z = 2x^2 + 2y^2$ is above $z = -2x + 3x^2 + 3y^2$.
(can test at a point like (1,0))

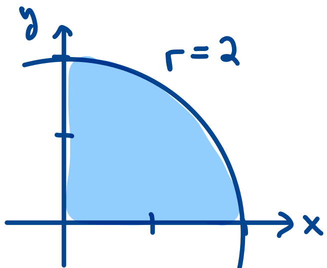
So integrand is $(2x^2 + 2y^2) - (-2x + 3x^2 + 3y^2)$

$\downarrow$ polar

$2r^2 - (-2r\cos\theta + 3r^2) = 2r\cos\theta - r^2$

Integral: $$\boxed{\int_{-\pi/2}^{\pi/2} \int_0^{2r\cos\theta} (2r\cos\theta - r^2)\, r \, dr \, d\theta}$$

6. *(13 points)* Find the center of mass of the lamina that occupies the quarter disk **in the first quadrant** bounded by $x^2+y^2=4$ with density function $\rho(x,y)=\underbrace{1+x^2+y^2}_{1+r^2}$.



Mass: $m=\int_0^{\pi/2}\int_0^2 (1+r^2)\,r\,dr\,d\theta$

$= \int_0^{\pi/2}\left(\frac{1}{2}r^2+\frac{1}{4}r^4\right)\Big]_{r=0}^{r=2} d\theta = \int_0^{\pi/2}(2+4)\,d\theta = 6\theta\Big]_0^{\pi/2} = 3\pi$

Note: region and density function are symmetric in x & y, so $\bar{x}=\bar{y}$.

$M_y = \int_0^{\pi/2}\int_0^2 (1+r^2)\overbrace{(r\cos\theta)}^{x}\,r\,dr\,d\theta = \int_0^{\pi/2}\cos\theta\left(\int_0^2 (r^2+r^4)\,dr\right)d\theta$

$= \int_0^{\pi/2}\cos\theta\left(\frac{1}{3}r^3+\frac{1}{5}r^5\right)\Big]_{r=0}^{r=2} d\theta = \int_0^{\pi/2}\cos\theta\left(\frac{8}{3}+\frac{32}{5}\right)d\theta$

$= \frac{136}{15}\int_0^{\pi/2}\cos\theta\,d\theta = \frac{136}{15}(\sin\theta)\Big]_0^{\pi/2} = \frac{136}{15}$

$x = \frac{\frac{136}{15}}{3\pi} = \frac{136}{45\pi}$, and $\bar{y}$ is the same!

$(\bar{x},\bar{y}) = \boxed{\left(\frac{136}{45\pi}, \frac{136}{45\pi}\right)}$

7. *(14 points)* Let $g(x) = \frac{1}{2}\ln(2x-3)$.

In this problem you will find the Taylor polynomial based at $b = 2$ (**NOT** at $b = 0$).

(a) Find the third Taylor polynomial, $T_3(x)$, for the function g based at $b = 2$.

$g(x) = \frac{1}{2}\ln(2x-3)$ $\quad g(2) = 0$

$g'(x) = \frac{1}{2x-3}$ $\quad g'(2) = 1$

$g''(x) = \frac{-2}{(2x-3)^2}$ $\quad g''(2) = -2$

$g'''(x) = \frac{8}{(2x-3)^3}$ $\quad g'''(2) = 8$

$T_3(x) = \boxed{(x-2) - (x-2)^2 + \frac{4}{3}(x-2)^3}$

(b) Use Taylor's inequality to find an upper bound (as sharp as possible) for $|f(x) - T_3(x)|$ on the interval $[1.8, 2.2]$.

$\left|g^{(4)}(x)\right| = \left|\frac{-48}{(2x-3)^4}\right|$ largest when denominator is smallest, at $x = 1.8$

$M = \frac{48}{(0.6)^4} = \frac{10000}{27}$

$|T_3(x) - f(x)| \le \frac{1}{24} \frac{10000}{27} (0.2)^4 = \frac{2}{81}$

≈ 0.02469

Upper bound: $\boxed{\frac{2}{81}}$

8. *(14 points)* Let $f(x) = x\sin(2x^3)$.

(a) Find the Taylor series for $f(x)$ based at 0.

Simplify your final answer and write it in sigma notation.

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

$\downarrow$ replace x w/ $2x^3$

$$\sin(2x^3) = \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k+1} x^{6k+3}}{(2k+1)!}$$

$\downarrow$ mult. by x

$$x\sin(2x^3) = \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k+1} x^{6k+4}}{(2k+1)!}$$

Taylor series: $\boxed{\sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k+1} x^{6k+4}}{(2k+1)!}}$

(b) Find $f^{(16)}(0)$.

Want x^{16} term of T. series. $6k+4=16 \rightarrow k=2$

This term is $\frac{2^5 x^{16}}{5!} = \frac{f^{(16)}(0)\,x^{16}}{16!}$ (by def'n of Taylor series)

$$\frac{2^5}{5!} = \frac{f^{(16)}(0)}{16!}$$

$f^{(16)}(0) =$ $\boxed{\frac{32 \cdot 16!}{5!}}$

(c) Give the first two nonzero terms of the Taylor series for $g(t) = \int_0^t x\sin(2x^3)\,dx$.

$$g(t) \approx \int_0^t \left(\frac{2x^4}{1} - \frac{8x^{10}}{6}\right)dx$$

$$= \left(\frac{2}{5}x^5 - \frac{4}{33}x^{11}\right)\Big]_0^t$$

First two nonzero terms: $\boxed{\frac{2}{5}t^5 - \frac{4}{33}t^{11}}$