1. *(12 points)* Find the equation for the plane that passes through the point $(1, 2, 4)$ and contains the line given by $x = 2t$, $y = 3 - t$, $z = 2 + 5t$. And give the x-intercept of this plane.

Plane contains $(1,2,4)$ & $(0,3,2)$: contains vector $\langle -1, 1, -2 \rangle$

Also contains dir. of line: $\langle 2, -1, 5 \rangle$.

Normal vector is orth. to both: $\langle -1, 1, -2 \rangle \times \langle 2, -1, 5 \rangle = \langle 3, 1, -1 \rangle$

Plane w/ normal vector $\langle 3, 1, -1 \rangle$ through $(1,2,4)$:

$$3(x-1) + (y-2) - (z-4) = 0$$

$$\downarrow$$

$$3x + y - z = 1$$

x-intercept: $\downarrow y = z = 0$

$$3x = 1 \rightarrow x = \frac{1}{3}$$

Plane Equation: $3x + y - z = 1$

x-intercept: $(x, y, z) =$ $\left(\frac{1}{3}, 0, 0\right)$

2. *(12 points)* Let a curve γ be the intersection of the plane $y+z=1$ and the cylinder $x^2+y^2=1$. Find the points where γ has maximum curvature.

$x = \cos t$
$y = \sin t$
$z = 1 - y = 1 - \sin t$

$\vec{r}(t) = \langle \cos t, \sin t, 1-\sin t \rangle$

$\vec{r}'(t) = \langle -\sin t, \cos t, -\cos t \rangle$

$\vec{r}''(t) = \langle -\cos t, -\sin t, \sin t \rangle$

$\vec{r}'(t) \times \vec{r}''(t) = \langle 0, 1, 1 \rangle$

$|\vec{r}'(t)| = \sqrt{\sin^2 t + \cos^2 t + \cos^2 t} = \sqrt{1+\cos^2 t}$

$|\vec{r}'(t) \times \vec{r}''(t)| = \sqrt{2}$

$K = \dfrac{\sqrt{2}}{(1+\cos^2 t)^{3/2}}$ largest when denom. is smallest $\rightarrow \cos^2 t = 0 \rightarrow t = \frac{\pi}{2}$ or $\frac{3\pi}{2}$

$\vec{r}(\frac{\pi}{2}) = \langle 0, 1, 0 \rangle$

$\vec{r}(\frac{3\pi}{2}) = \langle 0, -1, 2 \rangle$

Points: $(x, y, z) =$ $\boxed{(0,1,0) \ \& \ (0,-1,2)}$

3. *(6 points per part)* Parts (a) and (b), consider the surface implicitly defined by the equation

$$z(\cos(z)+2) = x^2 - y^2.$$

(a) Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

$\frac{\partial}{\partial x}$ $\frac{\partial}{\partial y}$

$$\frac{\partial z}{\partial x}(\cos z+2) - z\sin z\frac{\partial z}{\partial x} = 2x$$

$$\frac{\partial z}{\partial y}(\cos z+2) - z\sin z\frac{\partial z}{\partial y} = -2y$$

$$\frac{\partial z}{\partial x}(\cos z+2-z\sin z) = 2x$$

$$\frac{\partial z}{\partial y}(\cos z+2-z\sin z) = -2y$$

$\dfrac{\partial z}{\partial x} = \dfrac{2x}{\cos z+2-z\sin z}$

$\dfrac{\partial z}{\partial y} = \dfrac{2y}{\cos z+2-z\sin z}$

(b) Give the linear approximation, i.e. tangent plane, to the surface at the point where $(x, y) = (1, 1)$. And use this approximation to estimate the z-coordinate of a point on the surface where $(x, y) = (1.01, 0.98)$.

When $x=y=1 \rightarrow z(\cos z+2)=0 \rightarrow z=0$ ($\cos z+2$ is never 0)

$$\frac{\partial z}{\partial x} = \frac{2}{1+2} = \frac{2}{3} \qquad \frac{\partial z}{\partial y} = \frac{-2}{1+2} = \frac{-2}{3}$$

$$L(x,y) = \frac{2}{3}(x-1) - \frac{2}{3}(y-1) + 0$$

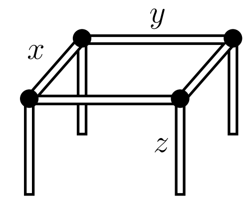
$$L(1.01, 0.98) = \frac{2}{3}(0.01) - \frac{2}{3}(-0.02) = 0.02$$

Linearization: $L(x,y) = \frac{2}{3}(x-1) - \frac{2}{3}(y-1)$

$z \approx$ 0.02

4. *(12 points)* Find the dimensions of a rectangular tent of maximum volume such that the sum of the lengths of its 8 posts (above the ground) is 36 m.

   To receive full credit, you must justify that your answer is indeed the maximum.

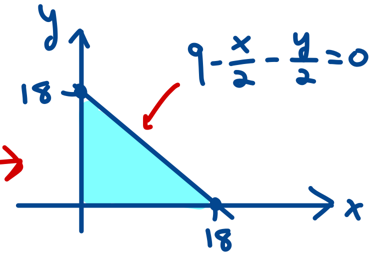


Want to maximize $V = xyz$

Constraint: $2x + 2y + 4z = 36 \rightarrow z = 9 - \frac{x}{2} - \frac{y}{2}$

Let $f(x,y) = V = xy\left(9 - \frac{x}{2} - \frac{y}{2}\right) = 9xy - \frac{1}{2}x^2y - \frac{1}{2}xy^2$

Domain: $x \geq 0$, $y \geq 0$, $z = 9 - \frac{1}{2}x - \frac{1}{2}y \geq 0$:



Note: on boundary, x, y, or z is zero, so volume $= 0$. So max cannot be on boundary! Must be at crit pt.

$f_x(x,y) = 9y - xy - \frac{1}{2}y^2 = 0$

$f_y(x,y) = 9x - \frac{1}{2}x^2 - xy = 0$

since $x \neq 0$ and $y \neq 0$ at max we can safely divide by x or y

$$9 - x - \frac{1}{2}y = 0$$
$$-2\left(9 - \frac{1}{2}x - y = 0\right)$$
$$-9 + \frac{3}{2}y = 0$$

$y = 6 \rightarrow 9 - x - 3 = 0 \rightarrow x = 6$

$z = 9 - 3 - 3 = 3$

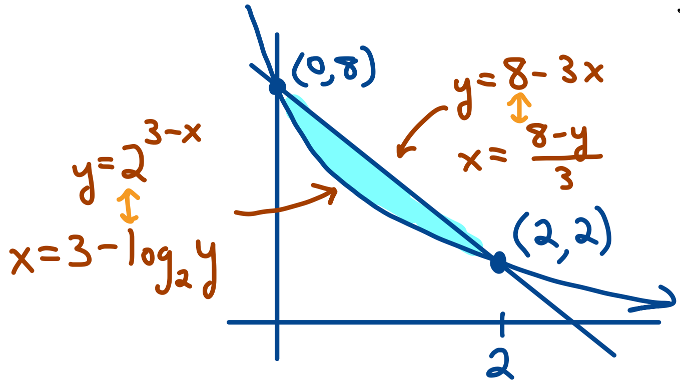
This is the only crit pt. not on boundary so it's the max.

Dimensions: $\boxed{x = 6 \text{ m} \quad y = 6 \text{m} \quad z = 3 \text{ m}}$

5. *(6 points per part)* For the following problems, set up an iterated integral as described.

You do not have to evaluate the integrals. Just set them up.

(a) Reverse the order of integration of $\displaystyle\int_0^2 \int_{2^{(3-x)}}^{8-3x} \sin(x^2)\, dy\, dx$.



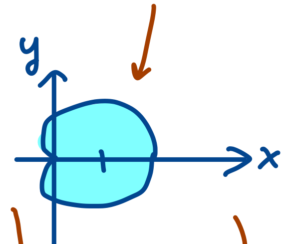
Integral: $\displaystyle\boxed{\int_2^8 \int_{3-\log_2 y}^{\frac{8-y}{3}} \sin(x^2)\, dx\, dy}$

(b) Write a double integral for the volume below the cone $z = \sqrt{x^2+y^2}$ and above the paraboloid $z = x^2 - x + y^2$.

(Hint: convert to polar!)

Intersection:

$\sqrt{x^2+y^2} = x^2 - x + y^2 \xrightarrow{\text{polar}} r = r^2 - r\cos\theta \rightarrow r = 1 + \cos\theta$



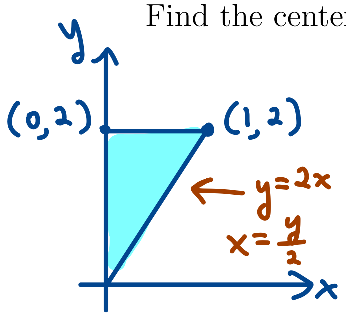
$\text{Height} = \sqrt{x^2+y^2} - (x^2 - x + y^2)$

$= r - r^2 + r\cos\theta$

Integral: $\displaystyle\boxed{\int_0^{2\pi} \int_0^{1+\cos\theta} (r - r^2 + r\cos\theta)\, r\, dr\, d\theta}$

6. *(12 points)* Let $\mathcal{L}$ be the lamina whose shape is the triangular region with vertices $(0,0)$, $(0,2)$, and $(1,2)$, and whose $\underbrace{\text{density at any point is proportional to its distance to the } y\text{-axis.}}$

Find the center of mass of $\mathcal{L}$.



$\rho(x,y) = k|x|$

(can just use $\rho(x,y)=x$ to find C.o.M. here)

Mass $\quad m = \int_0^2 \int_0^{y/2} x\,dx\,dy = \int_0^2 \left(\frac{1}{2}x^2\right)\Big]_{x=0}^{x=\frac{y}{2}} dy = \int_0^2 \frac{y^2}{8}\,dy = \frac{y^3}{24}\Big]_0^2 = \boxed{\frac{1}{3}}$

Moments $\quad M_y = \int_0^2 \int_0^{y/2} x^2\,dx\,dy = \int_0^2 \left(\frac{1}{3}x^3\right)\Big]_{x=0}^{x=\frac{y}{2}} = \int_0^2 \frac{y^3}{24}\,dy = \left(\frac{y^4}{96}\right)\Big]_0^2 = \boxed{\frac{1}{6}}$

$M_x = \int_0^2 \int_0^{y/2} xy\,dx\,dy = \int_0^2 \left(\frac{1}{2}x^2 y\right)\Big]_{x=0}^{x=\frac{y}{2}} dy = \int_0^2 \frac{y^3}{8}\,dy = \frac{y^4}{32}\Big]_0^2 = \boxed{\frac{1}{2}}$

Center of mass $= \left(\frac{M_y}{m}, \frac{M_x}{m}\right) = \left(\frac{\frac{1}{6}}{\frac{1}{3}}, \frac{\frac{1}{2}}{\frac{1}{3}}\right)$

$(\bar{x}, \bar{y}) = \boxed{\left(\frac{1}{2}, \frac{3}{2}\right)}$

7. For this problem, let $g(x) = (3+x)^{3/2}$ and consider the Taylor polynomials based at $b = 1$.

(a) *(5 points)* Find the second Taylor polynomial, $T_2(x)$, for the function g based at $b = 1$.

$g(x) = (3+x)^{3/2}$ $\quad g(1) = 8$

$g'(x) = \frac{3}{2}(3+x)^{1/2}$ $\quad g'(1) = 3$

$g''(x) = \frac{3}{4}(3+x)^{-1/2}$ $\quad g''(1) = \frac{3}{8}$

$T_2(x) = \boxed{8 + 3(x-1) + \frac{3}{16}(x-1)^2}$

(b) *(4 points)* Use $T_2(x)$ to approximate the value of $\sqrt{4.4^3}$

$\sqrt{4.4^3} = (3+1.4)^{3/2} = g(1.4)$

$g(1.4) \approx T_2(1.4) = 8 + 3(0.4) + \frac{3}{16}(0.4)^2 = 9.23$

$\sqrt{4.4^3} \approx \boxed{9.23}$

(c) *(5 points)* Use Taylor's inequality to find an upper bound for the error in your approximation in part (b).

$|g'''(x)| = \left|\frac{-3}{8}(3+x)^{-3/2}\right| = \frac{3}{8\sqrt{(3+x)^3}}$ largest when x is smallest

On interval $[1, 1.4]$, can use $x=1$:

$M = \frac{3}{8 \cdot 4^{3/2}} = \frac{3}{64}$

$\text{error} \le \frac{1}{6} \cdot \frac{3}{64} \cdot |1.4-1|^3 = 0.0005$

Upper bound: $\boxed{0.0005}$

8. For this problem, let $f(x) = \dfrac{16}{4-x^2} + 6\cos(2x)$

(a) *(6 points)* Give the Taylor series for f based at $b = 0$ using one sigma sign.

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$$

$$\downarrow \div 4$$

$$\frac{1}{4-4x} = \sum_{k=0}^{\infty} \frac{x^k}{4}$$

$$\downarrow x \to \frac{x^2}{4}$$

$$\frac{1}{4-x^2} = \sum_{k=0}^{\infty} \frac{x^{2k}}{4^{k+1}}$$

$$\downarrow \cdot 16$$

$$\frac{16}{4-x^2} = \sum_{k=0}^{\infty} \frac{x^{2k}}{4^{k-1}}$$

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

$$\downarrow x \to 2x$$

$$\cos(2x) = \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k} x^{2k}}{(2k)!}$$

$$\downarrow \cdot 6$$

$$6\cos(2x) = \sum_{k=0}^{\infty} \frac{6(-1)^k 2^{2k} x^{2k}}{(2k)!}$$

$$+$$

Taylor series: $\boxed{\displaystyle\sum_{k=0}^{\infty} \left(\frac{1}{4^{k-1}} + \frac{6(-1)^k 2^{2k}}{(2k)!}\right) x^{2k}}$

(b) *(3 points)* Find the largest open interval on which this Taylor series converges.

$\frac{1}{1-x}$ converges for $-1 < x < 1$

$$x \to x^2/4$$

$$-1 < \frac{x^2}{4} < 1$$

$$\downarrow$$

$$-4 < x^2 < 4$$

interval: $\boxed{(-2, 2)}$

(c) *(5 points)* Find $T_3(x)$, the third Taylor polynomial for f based at $b = 0$, then estimate the value of $\int_0^1 f(x)\,dx$ by replacing $f(x)$ with $T_3(x)$.

Taylor series from part (a): $\underbrace{\overbrace{10}^{k=0} - \overbrace{11x^2}^{k=1}}_{T_3(x)} + \overbrace{4.25x^4}^{k=2} + \ldots$ (ignore all terms w/ exponent > 3)

$$\int_0^1 f(x)\,dx \approx \int_0^1 (10 - 11x^2)\,dx = \left(10x - \frac{11}{3}x^3\right)\Big]_0^1 = \frac{19}{3}$$

$\int_0^1 f(x)\,dx \approx$ $\boxed{\frac{19}{3}}$