1. *(4 points per part)* **Parts (a), (b), and (c) are unrelated.**

   (a) The two lines $x = 1 + t$, $y = t$, $z = 1 - 2t$ and $x = 1$, $y = 4t$, $z = 1 - 3t$ intersect at the point $(1, 0, 1)$. Find the equation for the plane containing these two intersecting lines.

   $\langle 1, 1, -2\rangle \times \langle 0, 4, -3\rangle = \langle 5, 3, 4\rangle$

   plane w/ normal vector $\langle 5, 3, 4\rangle$ through pt $(1, 0, 1)$:

   $5(x-1) + 3y + 4(z-1) = 0$

   equation for plane: $\boxed{5x + 3y + 4z = 9}$

   (b) Find parameteric equations for the line of intersection of the two planes $x + y + z = 1$ and $2x + y - 3z = -3$.

   $x + y + z = 1$

   $-(2x + y - 3z = -3)$

   $-x + 4z = 4$

   One point: $x = 0$, $z = 1$, $y = 0$

   Another: $z = 0$, $x = -4$, $y = 5$

   Line through $(0,0,1)$ & $(-4,5,0)$:

   direction $\langle -4, 5, -1\rangle$

   equations for line: $\boxed{x = -4t \quad y = 5t \quad z = 1 - t}$

   (c) Find all points of intersection of the line through $(0, 0, 1)$ and $(3, 4, 1)$ and the paraboloid $100z = x^2 + y^2$.

   Line: $\left.\begin{aligned} x &= 3t \\ y &= 4t \\ z &= 1 \end{aligned}\right\}$

   Plug in: $100 = (3t)^2 + (4t)^2$

   $100 = 25t^2$

   $t = \pm 2$

   Plug back in

   intersection point(s): $(x, y, z) =$ $\boxed{(-6, -8, 1) \ \& \ (6, 8, 1)}$

2. *(6 points per part)* **Parts (a) and (b) are unrelated.**

(a) Find a vector function for the curve of intersection between the surface $5x^2 + y^2 - z^2 = 4$ and the plane $z = x$.

$5x^2 + y^2 - x^2 = 4$

$4x^2 + y^2 = 4 \rightarrow$ ellipse! Can parametrize w/ cost & sint.

$x = \cos t$, $y = 2\sin t$ works

$z = x = \cos t$

vector function: $\boxed{\vec{r}(t) = \langle \cos t, 2\sin t, \cos t \rangle}$

(b) Find the curvature of $\mathbf{r}(t) = \langle t^3, t^2 - 1, 3t + 7\rangle$ at the point $\underbrace{(-8, 3, 1)}$.

$\vec{r}\,'(t) = \langle 3t^2, 2t, 3 \rangle$ $\quad \vec{r}\,'(-2) = \langle 12, -4, 3 \rangle$ $\quad t = -2$

$\vec{r}\,''(t) = \langle 6t, 2, 0 \rangle$ $\quad \vec{r}\,''(-2) = \langle -12, 2, 0 \rangle$

$\vec{r}\,'(-2) \times \vec{r}\,''(-2) = \langle -6, -36, -24 \rangle$

$|\vec{r}\,'(-2)| = \sqrt{12^2 + 4^2 + 3^2} = 13$

$|\vec{r}\,'(-2)| = 6\sqrt{1^2 + 6^2 + 4^2} = 6\sqrt{53}$

$$K = \frac{|\vec{r}\,'(-2) \times \vec{r}\,''(-2)|}{|\vec{r}\,'(-2)|^3}$$

curvature: $K = \boxed{\dfrac{6\sqrt{53}}{13^3}}$

3. *(6 points per part)* For parts (a) and (b), let $f(x, y) = x^2y^2 - 4x^2 - y^2$.

   (a) Find all the saddle points of $f(x, y)$.

$f_x(x,y) = 2xy^2 - 8x = 0 \rightarrow 2x(y^2-4) = 0 \rightarrow x=0$ or $y = \pm 2$

$f_y(x,y) = 2x^2y - 2y = 0 \rightarrow 2y(x^2-1) = 0 \rightarrow y=0$ or $x = \pm 1$

Five crit. pts: $(0,0), (1,2), (-1,2), (1,-2), (-1,-2)$

$f_{xx}(x,y) = 2y^2 - 8$ $\quad D(0,0) > 0 \leftarrow$ not a saddle point

$f_{yy}(x,y) = 2x^2 - 2$ $\quad D(\pm 1, \pm 2) < 0$

$f_{xy}(x,y) = 4xy$

saddle points: $\boxed{(1,2), (1,-2), (-1,2), (-1,-2)}$

   (b) Find the equation of the tangent plane to the surface $z = f(x, y)$ at $x = 2, y = 1$ and use it to approximate $f(2.1, 0.8)$.

$f(2,1) = -13$

$f_x(2,1) = -12$

$f_y(2,1) = 6$

Tangent plane: $z = -13 - 12(x-2) + 6(y-1)$

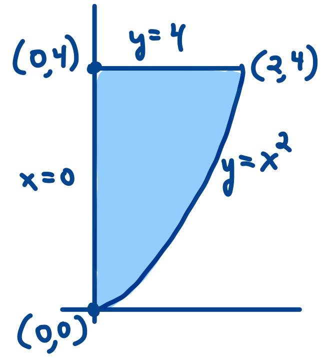
$f(2.1, 0.8) \approx -13 - 12(2.1-2) + 6(0.8-1) = -15.4$

tangent plane: $\boxed{z = -13 - 12(x-2) + 6(y-1)}$ $\quad f(2.1, 0.8) \approx \boxed{-15.4}$

4. *(12 points)* Find the absolute maximum and absolute minimum values of the function

$$f(x, y) = x^2 - y^2 + 2y$$

on the region enclosed by $x = 0$, $y = 4$ and $y = x^2$.



Crit pts: $f_x(x,y) = 2x = 0$, $f_y(x,y) = -2y + 2 = 0$ $\Big\}$ $x=0$, $y=1$

Boundary

Top: $y=4$: $f(x,4) = x^2 - 8$
$0 \le x \le 2$ $f'(x) = 2x = 0$
$x = 0$
Check $(0,4)$ & $(2,4)$

Left: $x=0$: $f(0,y) = -y^2 + 2y$
$0 \le y \le 4$ $f'(y) = -2y + 2 = 0$
$y = 1$
Check
$(0,0), (0,1), (0,4)$

Bottom-right: $f(x,x^2) = x^2 - x^4 + 2x^2$
$f'(x) = 6x - 4x^3 = 0$
$= 2x(3 - 2x^2) = 0$
$x = 0$, $x = \sqrt{\frac{3}{2}}$, $\underbrace{x = -\sqrt{\frac{3}{2}}}_{\text{not in domain}}$

Check $(0,0)$, $(2,4)$, $\left(\sqrt{\frac{3}{2}}, \frac{3}{2}\right)$

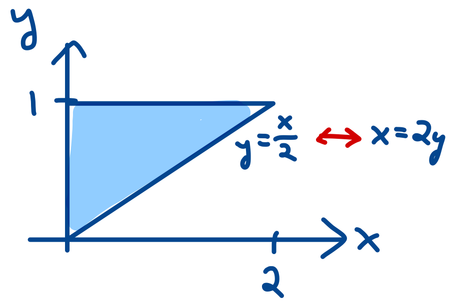
Points to check:

$f(0,1) = 1$
$f(0,4) = -8$
$f(2,4) = -4$
$f(0,0) = 0$
$f\left(\sqrt{\frac{3}{2}}, \frac{3}{2}\right) = \frac{9}{4}$

absolute extrema: min -8, max $\frac{9}{4}$

5. *(12 points)* Reverse the order of integration and evaluate

$$\int_0^2 \int_{x/2}^1 \frac{x}{y^3+1}\, dy dx.$$



$$= \int_0^1 \int_0^{2y} \frac{x}{y^3+1}\, dx\, dy$$

$$= \int_0^1 \left(\frac{\frac{1}{2}x^2}{y^3+1}\right)\Bigg]_{x=0}^{x=2y} dy$$

$$= \int_0^1 \frac{2y^3}{y^3+1}\, dy \qquad u = y^3+1 \quad du = 3y^2\,dy$$

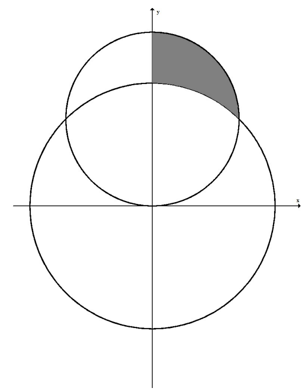
$$= \int_1^2 \frac{2}{3}\,\frac{1}{u}\, du = \frac{2}{3}\ln|u|\Bigg]_1^2$$

$$= \frac{2}{3}(\ln(2) - 0)$$

answer = $\boxed{\frac{2}{3}\ln(2)}$

6. *(12 points)* Let R be the region inside the circle $x^2 + y^2 = 4y$, outside the circle $x^2 + y^2 = 8$, and in the first quadrant (shown below). Evaluate

$$\iint_R \frac{x}{x^2+y^2}\, dA$$



Convert to polar:

$x^2+y^2=4y \to r^2=4r\sin\theta \to r=4\sin\theta$

$x^2+y^2=8 \to r=\sqrt{8}$

Intersection: $\sqrt{8}=4\sin\theta \to \sin\theta=\frac{\sqrt{2}}{2} \to \theta=\frac{\pi}{4}$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_{\sqrt{8}}^{4\sin\theta} \frac{\cancel{r}\cos\theta}{\cancel{r^2}}\,\cancel{r}\,dr\,d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos\theta\, r\Big]_{r=\sqrt{8}}^{r=4\sin\theta} d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(4\sin\theta\cos\theta - \sqrt{8}\cos\theta\right) d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(2\sin(2\theta) - \sqrt{8}\cos\theta\right) d\theta = \left(-\cos 2\theta - \sqrt{8}\sin\theta\right)\Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \left(1-\sqrt{8}\right) - \left(0-2\right)$$

$$= 3-\sqrt{8}$$

answer = $\boxed{3-\sqrt{8}}$

7. For parts (a)–(c), let $f(x) = \ln(2x-1)$.

(a) *(5 points)* Find the second Taylor polynomial, $T_2(x)$, for $f(x)$ based at $b = 1$.

$f(x) = \ln(2x-1)$ $\quad f(1) = 0$

$f'(x) = \frac{2}{2x-1}$ $\quad f'(1) = 2$

$f''(x) = \frac{-4}{(2x-1)^2}$ $\quad f''(1) = -4$

$T_2(x) = \boxed{2(x-1) - 2(x-1)^2}$

(b) *(4 points)* Use your answer to part (a) to approximate $\ln(1.1)$.

$\ln(1.1) = \ln(2.1-1) = f(1.05) \approx T_2(1.05)$

$= 2(1.05-1) - 2(1.05-1)^2 = 0.1 - 0.005$

$\ln(1.1) \approx \boxed{0.095}$

(c) *(5 points)* Find an upper bound (as sharp as possible) on the error for your answer from part (b).

$f'''(x) = \frac{16}{(2x-1)^3}$

max when x is smallest, at $x = 1$ on interval $[1, 1.05]$.

$\rightarrow$ Use $M = \frac{16}{1^3} = 16$

$|f(x) - T_2(x)| \leq \frac{1}{6}(16)(0.05)^3 = \frac{1}{3000}$

(Ok to instead use interval $[0.95, 1.05]$, giving $M = \frac{16}{.9^3}$, error ≤ 0.000457)

Error bound: $\boxed{\frac{1}{3000}} = 0.000\overline{3}$

8. For this problem, you may use the following basic Taylor series:

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k, \quad e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad \sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}, \quad \cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

(a) *(6 points)* Find the Taylor series for $f(x) = \int_0^x e^{2t^2}\,dt$ based at $b = 0$. Express your answer using $\sum$–notation.

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

replace x w/ $2t^2$

$$e^{2t^2} = \sum_{k=0}^{\infty} \frac{2^k t^{2k}}{k!}$$

$$\int_0^x e^{2t^2}\,dt = \sum_{k=0}^{\infty} \left(\int_0^x \frac{2^k t^{2k}}{k!}\,dt \right)$$

Taylor series: $\boxed{\sum_{k=0}^{\infty} \frac{2^k x^{2k+1}}{(2k+1)\,k!}}$

(b) *(3 points)* Find the open interval of convergence for the series you found in (a).

e^x converges for all x

None of these transformations change that.

Interval of convergence: $\boxed{(-\infty, \infty)}$

(c) *(5 points)* Find $f^{(2023)}(0)$, i.e. the 2023rd derivative of f at 0.

x^{2023} term of $\sum_{k=0}^{\infty} \frac{2^k x^{2k+1}}{(2k+1)\,k!}$ is at $2k+1 = 2023 \rightarrow k = 1011 \rightarrow \frac{2^{1011} x^{2023}}{2023 \cdot 1011!}$

$$\frac{f^{(2023)}(0)\, x^{2023}}{2023!} = \frac{2^{1011} x^{2023}}{2023 \cdot 1011!} \rightarrow f^{(2023)}(0) = \frac{2023!\, 2^{1011}}{2023 \cdot 1011!}$$

$f^{(2023)}(0) = \boxed{\frac{2022! \cdot 2^{1011}}{1011!}}$