1. (10 points) Each of the following multiple choice problems has one correct answer. Circle it. You do not need to show any reasoning.

(a) Suppose $\mathbf{comp_a b} = \frac{1}{2}|\mathbf{b}|$. Then the angle between $\mathbf{a}$ and $\mathbf{b}$ is...

(i) 30°. (ii) 45°. (iii) 60°. (iv) 90°.

$|\vec{b}|\cos\theta = \frac{1}{2}|\vec{b}| \longrightarrow \theta = 60°$

(b) Suppose $\mathcal{S}$ is the set of points P such that the distance from P to the x-axis is equal to 3. Then $\mathcal{S}$ is...

(i) a plane. (ii) a cylinder. (iii) a sphere. (iv) a cone.

$\sqrt{y^2+z^2} = 3 \rightarrow y^2+z^2=9$

(c) The surface $z = x^2 + 2xy$ is tangent to the plane $z = 6x + 4y - 8$ at the point...

(i) $(-2, 3, -8)$ (ii) $(0, 2, 0)$. (iii) $(2, 1, 8)$. (iv) $(4, 0, 16)$.

$\frac{\partial z}{\partial x} = 2x+2y, \ \frac{\partial z}{\partial y} = 2x$

$\frac{\partial z}{\partial x} = 6, \ \frac{\partial z}{\partial y} = 4$

$x=2, y=1, z=8$

(d) The value of $\int_2^5 \int_3^5 (5 + \sin^2(yx^2 + y^3))\, dy\, dx$ is between...

(i) 0 and 10. (ii) 10 and 20. (iii) 20 and 30. (iv) 30 and 40.

$5 \le 5+\sin^2(yx^2+y^3) \le 6$

base of solid is $3\times 2 \rightarrow$ area 6

$5\times 6 \le$ integral $\le 6\times 6$

(e) The Taylor series for $f(x) = \dfrac{1}{2-x^2}$ centered at $b = 0$ converges on the interval...

(i) $(-1, 1)$. (ii) $(-2, 2)$. (iii) $(-4, 4)$. (iv) $(-\sqrt{2}, \sqrt{2})$.

$\frac{1}{1-x}$ converges for $-1<x<1$

$\frac{1}{1-\frac{x^2}{2}}$ converges for $-1 < \frac{x^2}{2} < 1 \rightarrow -\sqrt{2} < x < \sqrt{2}$

$\frac{1}{2-x^2}$ also converges for $-\sqrt{2} < x < \sqrt{2}$

2. (12 pts) Let L be the line of intersection of the two planes

$$x + y + 2z = c \qquad \text{and} \qquad x - cy - cz = -1$$

where c is some real number. Find a value of c for which L is perpendicular to the plane $3x - y - z = 0$.

The direction vector of L is $\langle 1,1,2\rangle \times \langle 1,-c,-c\rangle$

$= \langle c, c+2, -c-1\rangle$

We want this to be parallel to $\langle 3,-1,-1\rangle$

So $\dfrac{c}{3} = \dfrac{c+2}{-1} = \dfrac{-c-1}{-1}$

$$\downarrow$$

$$\boxed{c = \frac{-3}{2}}$$

3. (12 pts) Find the curvature of the ellipse

$$x = 3\cos(t), \quad y = 4\sin(t), \quad z = 1,$$

at the points $\underbrace{(3,0,1)}_{t=0}$ and $\underbrace{(0,4,1)}_{t=\frac{\pi}{2}}$.

$\vec{r}(t) = \langle 3\cos(t), 4\sin(t), 1\rangle$

$\vec{r}\,'(t) = \langle -3\sin(t), 4\cos(t), 0\rangle$

$\vec{r}\,''(t) = \langle -3\cos(t), -4\sin(t), 0\rangle$

$t=0$ $\quad \vec{r}\,'(0) = \langle 0,4,0\rangle \quad \vec{r}\,''(0) = \langle -3,0,0\rangle \quad \vec{r}\,'(0)\times\vec{r}\,''(0) = \langle 0,0,12\rangle$

$$K = \frac{12}{4^3} = \boxed{\frac{3}{16}}$$

$t = \frac{\pi}{2}$ $\quad \vec{r}\,'(\frac{\pi}{2}) = \langle -3,0,0\rangle \quad \vec{r}\,''(\frac{\pi}{2}) = \langle 0,-4,0\rangle \quad \vec{r}\,'(\frac{\pi}{2})\times\vec{r}\,''(\frac{\pi}{2}) = \langle 0,0,12\rangle$

$$K = \frac{12}{3^3} = \boxed{\frac{4}{9}}$$

using $K = \dfrac{|\vec{r}\,'(t)\times\vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$

4. (14 pts) Find and classify all the critical points of $f(x, y) = 4xy - 3y + \frac{1}{x} - \frac{1}{4}\ln(y)$.
Clearly show your work in using the second derivative test and label your answers.

$f_x(x,y) = 4y - \frac{1}{x^2} = 0 \rightarrow 4y = \frac{1}{x^2}$

$f_y(x,y) = 4x - 3 - \frac{1}{4y} = 0$

$4x - 3 - x^2 = 0$

$-(x-1)(x-3) = 0$

$x=1$, $y=\frac{1}{4}$ or $x=3$, $y=\frac{1}{36}$ so $(1, \frac{1}{4})$ & $(3, \frac{1}{36})$

$f_{xx}(x,y) = \frac{2}{x^3}$ $f_{yy}(x,y) = \frac{1}{4y^2}$ $f_{xy}(x,y) = 4$

At $(1, \frac{1}{4})$ $D = (2)(4) - 4^2 = -8 < 0$

$\boxed{(1, \tfrac{1}{4}) \text{ gives a saddle point}}$

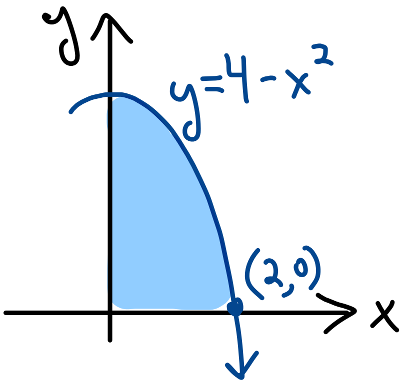
At $(3, \frac{1}{36})$. $D = \left(\frac{2}{27}\right)(324) - 4^2 = 8 > 0$

positive

so $\boxed{(3, \tfrac{1}{36}) \text{ gives a local minimum}}$

5. (14 pts) Compute the volume of the solid between the surface $x^2 + y + z = 4$ and the xy-plane above the first quadrant.

Intersection of $x^2+y+z=4$ and first quadrant in xy-plane



$z = 4-y-x^2$

$$\int_0^2 \int_0^{4-x^2} (4-y-x^2)\,dy\,dx$$

$$= \int_0^2 \left(4y - \frac{1}{2}y^2 - yx^2\right)\Big]_{y=0}^{y=4-x^2}$$

$$= \int_0^2 \left(4(4-x^2) - \frac{1}{2}(4-x^2)^2 - (4-x^2)x^2\right)dx$$

$$= \int_0^2 \left(\frac{1}{2}x^4 - 4x^2 + 8\right)dx$$

$$= \left(\frac{1}{10}x^5 - \frac{4}{3}x^3 + 8x\right)\Big]_0^2 = \frac{32}{10} - \frac{32}{3} + 16$$

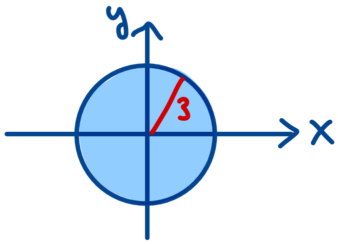
$$= \boxed{\frac{128}{15}}$$

6. (12 pts) Compute

$$\iint_R e^{-(x^2+y^2)}\, dA$$

where $R = \{(x, y) : x^2 + y^2 \le 9\}$.

Polar!



$$\int_0^{2\pi} \int_0^{3} e^{-r^2} r \, dr \, d\theta$$

$u = -r^2$

$du = -2r\,dr$

$$\int_0^{2\pi} \left(\int_0^{-9} \frac{-1}{2} e^u \, du \right) d\theta = \int_0^{2\pi} \left(\frac{-1}{2} e^u \right) \Bigg]_{u=0}^{u=-9} d\theta$$

$$= \int_0^{2\pi} \left(\frac{-1}{2} e^{-9} + \frac{1}{2} \right) d\theta$$

$$= \boxed{\pi \left(1 - e^{-9} \right)}$$

7. (12 pts) Let $f(x) = 1 + x + x^2 + 3x^3$.

   (a) Find the second-degree Taylor polynomial, $T_2(x)$, for $f(x)$ based at $b = 1$.

$f(x) = 1 + x + x^2 + 3x^3 \qquad f(1) = 6$

$f'(x) = 1 + 2x + 9x^2 \qquad f'(1) = 12$

$f''(x) = 2 + 18x \qquad f''(1) = 20$

$$\boxed{T_2(x) = 6 + 12(x-1) + 10(x-1)^2}$$

   (b) Determine $\overbrace{\text{an interval}}^{[1-a,\ 1+a]}$ around $b = 1$ on which

$$|T_2(x) - f(x)| < 0.024.$$

$f'''(x) = 18 = M$

$$\left| T_2(x) - f(x) \right| \leq \frac{1}{6}(18)|a|^3 < 0.024$$

$$|a|^3 < 0.008$$

$$|a| < 0.2$$

So $a = 0.1$ (for example) works.

$$\boxed{[0.9, 1.1]}$$

8. (14 pts) Let $f(x) = \dfrac{x^3}{1+x^4}$.

(a) Find the Taylor series for $f(x)$ based at zero. Express your answer using sigma notation.

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$$

$$\Big\downarrow \; x \to (-x^4)$$

$$\frac{1}{1+x^4} = \sum_{k=0}^{\infty} (-1)^k x^{4k}$$

$$\Big\downarrow \; \text{mult. by } x^3$$

$$\frac{x^3}{1+x^4} = \sum_{k=0}^{\infty} (-1)^k x^{4k+3}$$

(b) Use the Taylor series you found in (a) to find the Taylor series for

$$g(x) = x^2 \ln(1+x^4).$$

Express your answer using sigma notation.

Note: $\displaystyle\int \frac{x^3}{1+x^4}\,dx = \frac{1}{4}\ln\left|1+x^4\right| + C$

$u = 1+x^4$
$du = 4x^3\,du$

So $\displaystyle \ln\left(1+x^4\right) = 4\sum_{k=0}^{\infty} \int (-1)^k x^{4k+3}\,dx = \sum_{k=0}^{\infty} \frac{(-1)^k 4x^{4k+4}}{4k+4}$

$$\ln(1+x^4) = \underbrace{\sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+4}}{k+1}}_{\text{no } +C, \text{ because } \ln(1+0^4)=0}$$

$$x^2\ln\left(1+x^4\right) = \boxed{\sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+6}}{k+1}}$$