1. **Parts (a) and (b) are not related.**

(a) *(6 points)* Find parametric equations for the line through the point $(0, 3, 2)$ that is parallel to the plane $x + y + z = 1$ and perpendicular to the line $x = t, y = 3 - t, z = 2 + 2t$.

Direction vector of line is orthogonal to:

- Normal vector of $x+y+z=1$ $\langle 1,1,1\rangle$
- Direction vector of $x=t, y=3-t, z=2+2t$ $\langle 1,-1,2\rangle$

Direction is $\langle 1,1,1\rangle \times \langle 1,-1,2\rangle = \langle 3,-1,-2\rangle$

$x=3t$
$y=3-t$
$z=2-2t$

Parametric equations: $x=3t$, $y=3-t$, $z=2-2t$

(b) *(1.5 points each)* Consider the quadric surface $x^2 + 2z^2 = y^2$. For each of the following planes, identify the trace of the quadric surface in that plane.

Circle one answer for each part. You do not have to show any work.

i. $x = 1$: point line pair of lines circle ellipse parabola [✓ hyperbola] does not exist $1+2z^2=y^2$

ii. $y = 1$: point line pair of lines circle [✓ ellipse] parabola hyperbola does not exist $x^2+2z^2=1$

iii. $z = 0$: point line [✓ pair of lines] circle ellipse parabola hyperbola does not exist $x^2=y^2$

iv. $y = 0$: [✓ point] line pair of lines circle ellipse parabola hyperbola does not exist $x^2+2z^2=0$

2. *(12 points)* Find the equation of the tangent plane to the surface S at the point $P(4, -1, 5)$ given that the two curves

$$\mathbf{r}_1(t) = \left\langle 2t^2 + 3t + 5, 4t + 3, 4 - t^3 \right\rangle$$

and

$$\mathbf{r}_2(s) = \left\langle s^2, \frac{2}{s} - s, 2s + 1 \right\rangle$$

are on the surface S and they intersect at the point $P(4, -1, 5)$.

Give your answer in standard form: $Ax + By + Cz = D$.

$\vec{r}_1(t) = \langle 4, -1, 5 \rangle$ at $t = -1$

$\vec{r}_2(s) = \langle 4, -1, 5 \rangle$ at $s = 2$

Tangent vectors are both in tangent plane:

$\vec{r}_1{}'(t) = \langle 4t + 3, 4, -3t^2 \rangle \qquad \vec{r}_1{}'(-1) = \langle -1, 4, -3 \rangle$

$\vec{r}_2{}'(s) = \langle 2s, \frac{-2}{s^2} - 1, 2 \rangle \qquad \vec{r}_2{}'(2) = \langle 4, \frac{-3}{2}, 2 \rangle$

Normal vector: $\langle -1, 4, -3 \rangle \times \langle 4, \frac{-3}{2}, 2 \rangle = \langle \frac{7}{2}, -10, \frac{-29}{2} \rangle$

$$\frac{7}{2}(x-4) - 10(y+1) - \frac{29}{2}(z-5) = 0$$

$$7(x-4) - 20(y+1) - 29(z-5) = 0$$

$$7x - 20y - 29z = -97$$

Tangent plane (in standard form): $\boxed{7x - 20y - 29z = -97}$

3. *(12 points)* Find the linear approximation for

$$g(x, y, z) = 4\sqrt{x^2 + y^2} + 5\sqrt{4y^2 + z^2}$$

at the point $(3, 4, 6)$ and use it to approximate the value $g(2.97, 4.01, 6.04)$. Round your answer to three digits after the decimal.

$g_x(x,y,z) = \frac{4x}{\sqrt{x^2+y^2}}$ $\quad g_x(3,4,6) = 2.4$

$g_y(x,y,z) = \frac{4y}{\sqrt{x^2+y^2}} + \frac{20y}{\sqrt{4y^2+z^2}}$ $\quad g_y(3,4,6) = 11.2$

$g_z(x,y,z) = \frac{5z}{\sqrt{4y^2+z^2}}$ $\quad g_z(3,4,6) = 3$

$g(3,4,6) = 70$

$L(x,y,z) = 2.4(x-3) + 11.2(y-4) + 3(z-6) + 70$

$g(2.97, 4.01, 6.04) \approx L(2.97, 4.01, 6.04)$

$= 2.4(-0.03) + 11.2(0.01) + 3(0.04) + 70$

$= 70.16$

$L(x, y, z) =$ $2.4(x-3) + 11.2(y-4) + 3(z-6) + 70$

$g(2.97, 4.01, \underset{6}{\bullet}.04) \approx$ 70.16

4. *(12 points)* Find and classify all critical points for the function

$$f(x,y) = -\frac{1}{4}x^4 + \frac{2}{3}x^3 + 4xy - y^2.$$

$$f_x(x,y) = -x^3 + 2x^2 + 4y = 0$$

$$f_y(x,y) = 4x - 2y = 0 \rightarrow y = 2x$$

$$-x^3 + 2x^2 + 8x = 0$$

$$-x(x^2 - 2x - 8) = 0$$

$$-x(x-4)(x+2) = 0$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$x = 0$, $y = 0$ $\quad$ $x = 4$, $y = 8$ $\quad$ $x = -2$, $y = -4$

3 crit. pts: $(0,0)$, $(4,8)$, $(-2,-4)$

$$f_{xx}(x,y) = -3x^2 + 4x$$

$$f_{yy}(x,y) = -2$$

$$f_{xy}(x,y) = 4$$

$D(0,0) = 0 - 16 < 0$ → saddle pt

$D(4,8) = 64 - 16 > 0$, $f_{xx}, f_{yy} < 0$ → local max

$D(-2,-4) = 40 - 16 > 0$, $f_{xx}, f_{yy} < 0$ → local max

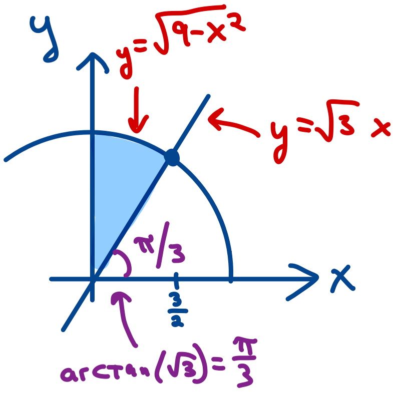
Points **and** classification:

saddle pt @ $(0,0)$

local max @ $(4,8)$

local max @ $(-2,-4)$

5. *(12 points)* Evaluate the integral by converting to polar coordinates:



$$\int_0^{3/2}\int_{(\sqrt{3})x}^{\sqrt{9-x^2}} \underbrace{2xy}_{2r\cos\theta\, r\sin\theta}\,dy\,dx$$

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\int_0^3 2r^3\cos\theta\sin\theta\,dr\,d\theta$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos\theta\sin\theta\left(\frac{1}{2}r^4\right)\Big]_{r=0}^{r=3} d\theta$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{81}{2}\sin\theta\cos\theta\,d\theta$$

$u=\sin\theta$

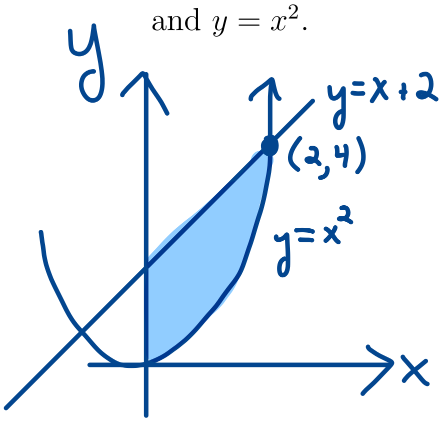
$du=\cos\theta\,d\theta$

$$= \int_{\frac{\sqrt{3}}{2}}^{1} \frac{81}{2}u\,du = \frac{81}{4}u^2\Big]_{\frac{\sqrt{3}}{2}}^{1}$$

$$= \frac{81}{4}\left(1-\frac{3}{4}\right) = \frac{81}{16}$$

Answer: $\boxed{\frac{81}{16}}$

6. *(12 points)* Find the y-coordinate $\bar{y}$ of the center of mass of the lamina with density function $\rho(x, y) = x$ that occupies the region D **in the first quadrant** bounded by the curves $y = x+2$ and $y = x^2$.



$$m = \int_0^2 \int_{x^2}^{x+2} x \, dy \, dx = \int_0^2 \left(yx\right)\Big]_{y=x^2}^{y=x+2} dx = \int_0^2 \left(x(x+2) - x^3\right) dx$$

$$= \int_0^2 \left(x^2 + 2x - x^3\right) dx = \left(\frac{1}{3}x^3 + x^2 - \frac{1}{4}x^4\right)\Big]_0^2 = \frac{8}{3} + 4 - 4 = \frac{8}{3}$$

$$M_x = \int_0^2 \int_{x^2}^{x+2} yx \, dy \, dx = \int_0^2 \left(\frac{1}{2}y^2 x\right)\Big]_{y=x^2}^{y=x+2} dx = \int_0^2 \left(\frac{1}{2}(x+2)^2 x - \frac{1}{2}x^5\right) dx$$

$$= \frac{1}{2}\int_0^2 \left(x^3 + 4x^2 + 4x - x^5\right) dx$$

$$= \frac{1}{2}\left(\frac{1}{4}x^4 + \frac{4}{3}x^3 + 2x^2 - \frac{1}{6}x^6\right)\Big]_0^2$$

$$= \frac{1}{2}\left(4 + \frac{32}{3} + 8 - \frac{64}{6}\right) = 6$$

$$\bar{y} = \frac{M_x}{m} = \frac{6}{\left(\frac{8}{3}\right)} = \frac{9}{4} = 2.25$$

$\bar{y} = \boxed{2.25}$

7. For this problem, let $f(x) = (2x-1)^{5/2}$.

(a) *(5 points)* Find the second Taylor polynomial, $T_2(x)$, for the function f based at $b = 1$.

$f(x) = (2x-1)^{5/2}$ $\quad f(1) = 1$

$f'(x) = 5(2x-1)^{3/2}$ $\quad f'(1) = 5$

$f''(x) = 15(2x-1)^{1/2}$ $\quad f''(1) = 15$

$T_2(x) = \boxed{1 + 5(x-1) + \frac{15}{2}(x-1)^2}$

(b) *(4 points)* Use $T_2(x)$ to approximate the value of $1.04^{5/2}$.

$1.04^{5/2} = (2x-1)^{5/2}$

↓

$1.04 = 2x-1$

$x = 1.02$

$1.04^{5/2} = f(1.02) \approx T_2(1.02)$

$= 1 + 5(.02) + \frac{15}{2}(.02)^2 = 1.103$

$1.04^{5/2} \approx \boxed{1.103}$

(c) *(5 points)* Use Taylor's inequality to find an upper bound for the error in your approximation in part (b).

$f'''(x) = \frac{15}{2\sqrt{2x-1}}$ on $[1, 1.02]$, this is greatest when $x = 1$

$M = \frac{15}{2}$

$|f(x) - T_2(x)| \leq \frac{1}{6} \frac{15}{2} (.02)^3 = 0.00001$

(larger bounds ok w/ justification)

Upper bound: $\boxed{0.00001}$

8. For this problem, let $f(x) = \dfrac{x^4}{1+3x^6} + x\sin(2x^3)$

(a) *(6 points)* Give the Taylor series for f based at $b = 0$ using one sigma sign.

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$$

$\downarrow x \to -3x^6$

$$\frac{1}{1+3x^6} = \sum_{k=0}^{\infty} (-1)^k 3^k x^{6k}$$

$\downarrow \cdot x^4$

$$\frac{x^4}{1+3x^6} = \sum_{k=0}^{\infty} (-1)^k 3^k x^{6k+4}$$

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

$\downarrow x \to 2x^3$

$$\sin(2x^3) = \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k+1} x^{6k+3}}{(2k+1)!}$$

$\downarrow \cdot x$

$$x\sin(2x^3) = \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k+1} x^{6k+4}}{(2k+1)!}$$

Taylor series: $\boxed{\sum_{k=0}^{\infty} (-1)^k \left(3^k + \frac{2^{2k+1}}{(2k+1)!}\right) x^{6k+4}}$

(b) *(3 points)* Find the largest open interval on which this Taylor series converges.

$\sum_{k=0}^{\infty} x^k$ converges for $-1 < x < 1$

$\downarrow x \to -3x^6$

$-1 < -3x^6 < 1$

$-\frac{1}{3} < x^6 < \frac{1}{3}$

Other transformations are irrelevant (T. series for $\sin x$ converges everywhere)

Interval: $\boxed{\left(\sqrt[6]{\frac{-1}{3}}, \sqrt[6]{\frac{1}{3}}\right)}$

(c) *(5 points)* Find $f^{(1000)}(0)$ (i.e. the 1000th derivative of f at 0).

Need x^{1000} term of $\sum_{k=0}^{\infty} (-1)^k \left(3^k + \frac{2^{2k+1}}{(2k+1)!}\right) x^{6k+4}$ $\rightarrow$ $6k+4 = 1000$

$6k = 996$

$k = 166$

It's $\left(3^{166} + \frac{2^{333}}{333!}\right) x^{1000}$.

By def'n of T. series, $3^{166} + \frac{2^{333}}{333!} = \frac{f^{(1000)}(0)}{1000!}$

$\downarrow$

$f^{(1000)}(0) = \boxed{1000!\left(3^{166} + \frac{2^{333}}{333!}\right)}$