1. *(12 points)* Let $\mathcal{P}$ be the plane given by the equation

$$-x + y + z = 2.$$

The point $Q(2, 1, -1)$ is not on the plane $\mathcal{P}$. Let L_1 be the line through Q and orthogonal to $\mathcal{P}$, and let R be the point of intersection of L_1 and $\mathcal{P}$. Let W be another point $(-1, 3, 1)$.

Find parametric equations of the line L_2 that passes through points W and R.

L_1 has direction $\underbrace{\langle -1, 1, 1 \rangle}_{\text{normal vector of } P}$ and passes through $(2,1,-1)$.

L_1: $x = 2 - t$, $y = 1 + t$, $z = -1 + t$

Intersection of L_1 with P:

$-(2-t) + (1+t) + (-1+t) = 2$

$\downarrow$

$-2 + 3t = 2$

$t = \frac{4}{3}$

$R = \left(\frac{2}{3}, \frac{7}{3}, \frac{1}{3}\right)$

$W = (-1, 3, 1)$

$\overrightarrow{RW} = \left\langle \frac{-5}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle$

(can use $\langle -5, 2, 2 \rangle$ as direction)

L_2: $x = -1 - 5t$
$y = 3 + 2t$
$z = 1 + 2t$

Parametric equations: ______________________

2. *(6 points per part)* **Parts (a) and (b) of this question are unrelated.**

   (a) Find the equation for the quadric surface consisting of all points that are equidistant to the plane $y = 3$ and the point $(0, 1, 2)$. Then identify the quadric surface.

$$|y-3| = \sqrt{x^2+(y-1)^2+(z-2)^2}$$

$$(y-3)^2 = x^2+(y-1)^2+(z-2)^2$$

$$y^2-6y+9 = x^2+y^2-2y+1+(z-2)^2$$

Equation for surface: $-4(y-2) = x^2+(z-2)^2$

Name of surface: Circular paraboloid (or elliptical)

   (b) Let P be the point where $\mathbf{r}(t) = \left\langle t^2+t, t^3, \frac{-3}{2}t^2 - 2t \right\rangle$ intersects the plane $y = -8$.

   Find the curvature of $\mathbf{r}(t)$ at P.

$$t^3 = -8$$
$$t = -2$$

$$\vec{r}\,'(t) = \langle 2t+1, 3t^2, -3t-2\rangle$$

$$\vec{r}\,''(t) = \langle 2, 6t, -3\rangle$$

$$\vec{r}\,'(-2) = \langle -3, 12, 4\rangle$$

$$\vec{r}\,''(-2) = \langle 2, -12, -3\rangle$$

$$\vec{r}\,'(-2)\times\vec{r}\,''(-2) = \langle 12, -1, 12\rangle$$

$$|\vec{r}\,'(-2)| = \sqrt{9+144+16} = 13$$

$$|\vec{r}\,'(-2)\times\vec{r}\,''(-2)| = \sqrt{144+1+144} = 17$$

$K =$

Curvature: $\dfrac{17}{13^3}$

3. *(12 points)* Find the equation of the plane tangent to the surface

$$z = x^2 y - \sqrt{3x - y} - \frac{x}{y}$$

at the point $(6, 2, 65)$. **Write your answer in the form** $Ax + By + Cz = D$**.**

$$\frac{\partial z}{\partial x} = 2xy - \frac{3}{2\sqrt{3x-y}} - \frac{1}{y}$$

$$\hookrightarrow \text{at } (6,2,65), \ \frac{\partial z}{\partial x} = 24 - \frac{3}{8} - \frac{1}{2} = \frac{185}{8}$$

$$\frac{\partial z}{\partial y} = x^2 + \frac{1}{2\sqrt{3x-y}} + \frac{x}{y^2}$$

$$\hookrightarrow \text{at } (6,2,65), \ \frac{\partial z}{\partial y} = 36 + \frac{1}{8} + \frac{3}{2} = \frac{301}{8}$$

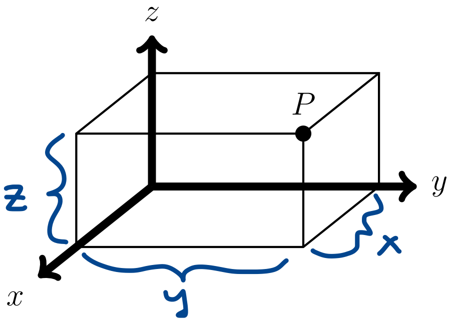
$$z = \frac{185}{8}(x-6) + \frac{301}{8}(y-2) + 65$$

$$8z = 185(x-6) + 301(y-2) + 520$$

Tangent plane equation: $185x + 301y - 8z = 1192$

4. *(12 points)* Find the volume of the largest rectangular box in the first octant with three faces in the coordinate planes and the vertex opposite the origin (i.e. the one marked P in the picture below) on the surface $z = 20 - x^2 - y$.

In order to receive full credit, you must show that your answer really is the maximum.



Volume $= xyz$ $\quad z = 20 - x^2 - y$

Volume $= f(x,y) = xy(20 - x^2 - y)$

Want abs. max of $f(x,y) = 20xy - x^3y - xy^2$

$f_x(x,y) = 20y - 3x^2y - y^2 = 0 \rightarrow y(20 - 3x^2 - y) = 0$

$f_y(x,y) = 20x - x^3 - 2xy = 0 \rightarrow x(20 - x^2 - 2y) = 0$

max can't occur at $x=0$ or $y=0$ (volume would $=0$),

so $\quad 20 - 3x^2 - y = 0$

$-3(\ 20 - x^2 - 2y = 0\ )$

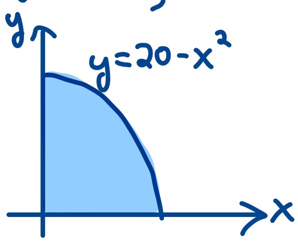
$-40 + 5y = 0$

$y = 8$

$20 - 3x^2 - 8 = 0 \rightarrow 12 = 3x^2 \rightarrow x = 2 \qquad z = 20 - x^2 - y = 8$

Only critical point (with nonzero volume) is $x=2$, $y=8$.

Furthermore, the domain is this region and the boundary is where



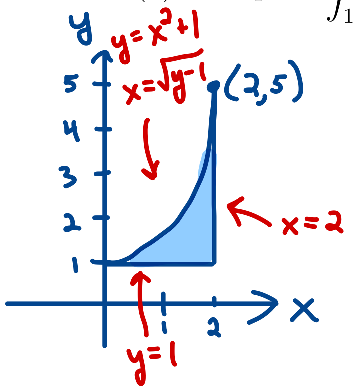
$x, y,$ or $z = 0$. So the max can't occur on the boundary.

Therefore the only possible max is when $x=2$, $y=8$, $z=8$

Maximum volume: 128

5. *(7 points per part)* **Part (a) and (b) of this question are not related.**

(a) Compute $\displaystyle\int_1^5 \int_{\sqrt{y-1}}^2 \frac{x^2}{1+x^5}\,dx\,dy$ (Hint: reverse the order of integration)



$$\int_0^2 \int_1^{x^2+1} \frac{x^2}{1+x^5}\,dy\,dx = \int_0^2 \left(\frac{y x^2}{1+x^5} \right]_{y=1}^{y=x^2+1} dx$$

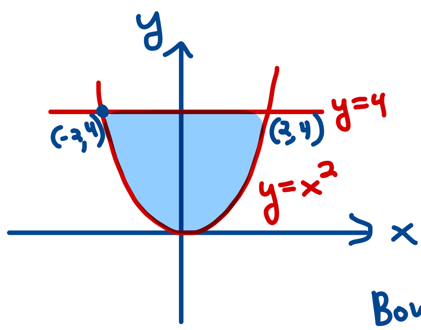
$$= \int_0^2 \frac{x^4}{1+x^5}\,dx = \frac{1}{5}\int_1^{33} \frac{1}{u}\,du = \frac{1}{5}\ln|u| \Big]_1^{33}$$

$u = 1+x^5$
$du = 5x^4\,dx$

$$= \frac{1}{5}\ln(33) - \underbrace{\frac{1}{5}\ln(1)}_{0}$$

Answer = $\frac{1}{5}\ln(33)$

(b) Set up (but **do not evaluate**) an integral for the volume of the solid enclosed by the cylinder $y = x^2$ and the planes $y + z = 4$ and $z = 0$.



intersect at $y=4$

height at (x,y) is $z = 4-y$

Bounds: $-2 \le x \le 2$
$x^2 \le y \le 4$

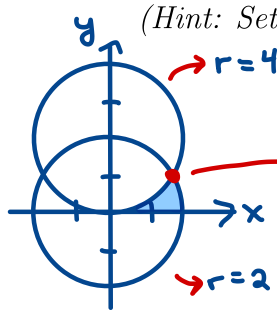
Volume = $\displaystyle\int_{-2}^{2} \int_{x^2}^{4} \boxed{4-y}\; d\,y\; d\,x$

6. *(6 points per part)* A lamina D occupies the region **in the first quadrant** that lies inside the circle $x^2 + y^2 = 4$ and outside the circle $x^2 + (y-2)^2 = 4$.

   The density at any point is $\rho(x, y) = \dfrac{x}{\sqrt{x^2 + y^2}}$.

   (a) Find the mass of D.

   (Hint: Set up and evaluate a double integral in polar coordinates.)



Intersection when $4\sin\theta = 2$

$\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}$

Want $\displaystyle\iint_D \frac{x}{\sqrt{x^2+y^2}}\,dA$ $x \to r\cos\theta$, $\sqrt{x^2+y^2} \to r$

$$\text{Mass} = \int_0^{\pi/6}\int_{4\sin\theta}^{2} \frac{\cancel{r}\cos\theta}{\cancel{r}}\, r\,dr\,d\theta = \int_0^{\pi/6}\left(\frac{1}{2}\cos\theta\, r^2\right)\Bigg]_{r=4\sin\theta}^{r=2} d\theta = \int_0^{\pi/6}\frac{1}{2}\cos\theta\left(4 - 16\sin^2\theta\right)d\theta$$

$u = \sin\theta$

$du = \cos\theta\, d\theta$

$$= \int_0^{1/2}\left(2 - 8u^2\right)du = \left(2u - \frac{8}{3}u^3\right)\Bigg]_0^{1/2} = 1 - \frac{1}{3}$$

Mass = $\frac{2}{3}$

   (b) Find $\overline{y}$, which is the y-coordinate of the center of mass.

$$M_x = \iint_D \frac{xy}{\sqrt{x^2+y^2}}\,dA = \int_0^{\pi/6}\int_{4\sin\theta}^{2} \frac{r^2\cos\theta\sin\theta}{\cancel{r}}\,\cancel{r}\,dr\,d\theta$$

$$= \int_0^{\pi/6}\left(\frac{1}{3}\sin\theta\cos\theta\, r^3\right)\Bigg]_{r=4\sin\theta}^{r=2} = \int_0^{\pi/6}\left(\frac{1}{3}\sin\theta\cos\theta\left(8 - 64\sin^3\theta\right)d\theta\right.$$

$u = \sin\theta$

$du = \cos\theta\, d\theta$

$$= \int_0^{1/2}\frac{1}{3}u\left(8 - 64u^3\right)du = \frac{1}{3}\left(4u^2 - \frac{64}{5}u^5\right)\Bigg]_0^{1/2} = \frac{1}{3}\left(1 - \frac{2}{5}\right) = \frac{1}{5}$$

$$\bar{y} = \frac{M_x}{m} = \frac{\frac{1}{5}}{\frac{2}{3}}$$

$\overline{y}$ = $\frac{3}{10}$

7. *(6 points per part)* Let $f(x) = \sin(x-1) + x^2 + e^x$.

(a) Find the second Taylor polynomial $T_2(x)$ for f based at $b = 1$.

$f(1) = 1+e$

$f'(x) = \cos(x-1) + 2x + e^x \qquad f'(1) = 3+e$

$f''(x) = -\sin(x-1) + 2 + e^x \qquad f''(1) = 2+e$

$T_2(x) = (1+e) + (3+e)(x-1) + \frac{1}{2}(2+e)(x-1)^2$

$T_2(x) =$ ________________

(b) Find an upper bound for $|f(x) - T_2(x)|$ on the interval $\left[\frac{1}{2}, \frac{3}{2}\right]$.

In order to receive credit, you must justify your answer.

$f'''(x) = \underbrace{-\cos(x-1)}_{|\cos(x-1)| \leq 1} + \underbrace{e^x}_{\leq e^{3/2}}$ on $\left[\frac{1}{2}, \frac{3}{2}\right]$

So one possible M is $1 + e^{3/2}$.

$|f(x) - T_2(x)| \leq \frac{1}{6}(1 + e^{3/2})\left|\frac{3}{2} - 1\right|^3$

Upper bound = $\frac{1}{48}(1 + e^{3/2})$

8. For this problem, you may use the following basic Taylor series:

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k, \quad e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad \sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}, \quad \cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

(a) *(6 points)* Use the list of Basic Series to find the Taylor series for $f(x) = x^2 \arctan(-3x)$ based at $b = 0$. (Use Σ notation.)

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$$

$$\downarrow x \to -x^2$$

$$\frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-1)^k x^{2k}$$

$$\downarrow \int$$

$$\arctan(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}$$

$$x \to -3x$$

$$\arctan(-3x) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} 3^{2k+1} x^{2k+1}}{2k+1}$$

$$\downarrow \cdot x^2$$

$$x^2 \arctan(-3x) =$$

Taylor series: $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^{k+1} 3^{2k+1} x^{2k+3}}{2k+1}$

(b) *(3 points)* Find the open interval on which the series in part (a) converges.

$\frac{1}{1-x}$ converges for $-1 < x < 1$

Substitutions:

$x \to -x^2$

$-1 < -x^2 < 1$

$x \to -3x$

$-1 < -(-3x)^2 < 1$

$-1 < 9x^2 < 1$

$\frac{-1}{9} < x^2 < \frac{1}{9}$

$\frac{-1}{3} < x < \frac{1}{3}$

Interval of convergence: $\left(\frac{-1}{3}, \frac{1}{3}\right)$

(c) *(5 points)* Find $f^{2023}(0)$. (That is, the 2023th derivative of f at 0.) Give an exact answer.

x^{2023} term of Taylor series is where $2k+3=2023 \to k=1010$

Term is $\dfrac{-3^{2021} x^{2023}}{2021} = \dfrac{f^{(2023)}(0)}{2023!} x^{2023}$

$f^{(2023)}(0) = \dfrac{-3^{2021} \cdot 2023!}{2021}$