1. (*4 points per part*) For parts (a)–(c), let $\mathcal{P}$ be the plane $8x - 4y + z = 3$.

(a) Find the acute angle between $\mathcal{P}$ and the plane $x = 4$.

Normal vector of P: $\langle 8, -4, 1 \rangle$

Normal vector of $x=4$: $\langle 1, 0, 0 \rangle$

$$\underbrace{\langle 8,-4,1\rangle \cdot \langle 1,0,0\rangle}_{8} = \underbrace{|\langle 8,-4,1\rangle|}_{9}\underbrace{|\langle 1,0,0\rangle|}_{1}\cos(\theta)$$

$$8 = 9 \quad 1$$

$$\theta = \boxed{\cos^{-1}\left(\frac{8}{9}\right)}$$

≈ 0.476 rad

$\approx 27.3°$

(b) Write parametric equations for the line of intersection between $\mathcal{P}$ and the $\underbrace{xz\text{-plane}}_{y=0}$.

$8x + z = 3$

$z = 3 - 8x$

$$\boxed{\begin{aligned} x &= t \\ y &= 0 \\ z &= 3 - 8t \end{aligned}}$$

(c) Find the point on the plane $\mathcal{P}$ that is closest to $(26, -7, 10)$.

Line through $(26, -7, 10)$ normal to P: $x = 26 + 8t$, $y = -7 - 4t$, $z = 10 + t$

Intersection w/ P:

$8(26 + 8t) - 4(-7 - 4t) + 10 + t = 3$

$208 + 64t + 28 + 16t + 10 + t = 3$

$81t = -243$

$t = -3$

$$\boxed{(2, 5, 7)}$$

2. *(9 points)* The acceleration of a particle at time t is given by the vector function

$$\mathbf{a}(t) = \langle 2t, 4, 45\sqrt{t}\rangle \text{ m/s}^2.$$

The particle is in the same position at time $t = 1$ as it is at time $t = 4$.

What is the **speed** of the particle at time $t = 0$?

$$\vec{v}(t) = \left\langle t^2 + C_1,\ 4t + C_2,\ 30t^{3/2} + C_3 \right\rangle$$

$$\vec{r}(t) = \left\langle \tfrac{1}{3}t^3 + C_1 t + C_4,\ 2t^2 + C_2 t + C_5,\ 12t^{5/2} + C_3 t + C_6 \right\rangle$$

$$\vec{r}(1) = \left\langle \tfrac{1}{3} + C_1 + C_4,\ 2 + C_2 + C_5,\ 12 + C_3 + C_6 \right\rangle$$

$$\updownarrow \qquad \updownarrow \qquad \updownarrow$$

$$\vec{r}(4) = \left\langle \tfrac{64}{3} + 4C_1 + C_4,\ 32 + 4C_2 + C_5,\ 384 + 4C_3 + C_6 \right\rangle$$

$$\tfrac{1}{3} + C_1 = \tfrac{64}{3} + 4C_1 \qquad 2 + C_2 = 32 + 4C_2 \qquad 12 + C_3 = 384 + 4C_3$$

$$-3C_1 = 21 \qquad -3C_2 = 30 \qquad -3C_3 = 372$$

$$C_1 = -7 \qquad C_2 = -10 \qquad C_3 = -124$$

$$\vec{v}(0) = \langle C_1, C_2, C_3 \rangle = \langle -7, -10, -124 \rangle$$

$$\text{speed} = \left| \langle -7, -10, -124 \rangle \right| = \sqrt{7^2 + 10^2 + 124^2} = \boxed{\sqrt{15525} \text{ m/s}}$$

$$\approx 124.6 \text{ m/s}$$

3. *(11 points)* Let $f(x, y) = 2x^2 - 2xy + y^2 - 3y$.

(a) Find and classify the critical points of f.

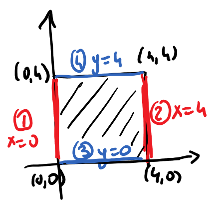
$$\begin{cases} f_x = 4x - 2y = 0 \\ f_y = -2x + 2y - 3 = 0 \end{cases} \Rightarrow \begin{cases} y = 2x \\ -2x + 2(2x) - 3 = 0 \end{cases} \Rightarrow 2x = 3 \Rightarrow x = 3/2$$

$\boxed{\text{Only one CP: } (3/2, 3)}$

$D = f_{xx} f_{yy} - (f_{xy})^2 = (4)(2) - (-2)^2 = 8 - 4 = 4 > 0 \Rightarrow$ local optima

$f_{xx} = 4 > 0 \Rightarrow \boxed{\text{local minimum}}$

(b) Find the global minimum and global maximum of f on the square $0 \le x \le 4$, $0 \le y \le 4$.



Ⅰ CP's inside the region: $(3/2, 3)$

$z = 2(3/2)^2 - 2(3/2)(3) + \cancel{3^2} - \cancel{3(3)} = 9/2 - 18/2 = -9/2$

Ⅱ CP's on boundary.

$\boxed{① \; x = 0}$ $(0 \le y \le 4)$: $g_1(y) = y^2 - 3y \Rightarrow g_1'(y) = 2y - 3 \Rightarrow$ CP $y = 3/2 \le 4$ ✓

so candidate $(0, 3/2)$ w/ $z = g(3/2) = 9/4 - 9/2 = -9/4$

$\boxed{② \; x = 4}$ $(0 \le y \le 4)$: $g_2(y) = y^2 - 11y + 32 \Rightarrow g_2'(y) = 2y - 11 \Rightarrow y = \frac{11}{2} > 4$ ✗

discard: no CP within domain $0 \le y \le 4$.

$\boxed{③ \; y = 0}$: $g_3(x) = 2x^2 \Rightarrow g_3'(x) = 4x \Rightarrow$ CP: $(0,0)$ ✓

candidate $(0,0) \Rightarrow z = 0$

$\boxed{④ \; y = 4}$: $g_4(x) = 2x^2 - 8x + 16 - 12 = 2x^2 - 8x + 4$ $\quad 2(4) - 8(2) + 4 = 8 - 16 + 4$

$g_4'(x) = 4x - 8 \Rightarrow$ CP: $x = 2 \Rightarrow$ candidate $(2, 4)$

Ⅲ corners: $(0,0) \to z = 0$ $\quad (4,0) \to z = 2(4)^2 = 32$

$(0,4) \to z = 16 - 12 = 4$ $\quad (4,4) \to z = 2(4)^2 - 2(4)(4) + 4^2 - 3(4) = 4$

Candidates:

	(x, y)	$z = f(x,y)$
CP's inside square	$(3/2, 3)$	$-9/2$
CP on $x=0$:	$(0, 3/2)$	$-9/4$
CP on $y=0$: (& corner)	$(0,0)$	0
CP on $y=4$:	$(2,4)$	-4
Corners:	$(0,4)$	4
	$(4,0)$	32
	$(4,4)$	4

Answer: $\boxed{\text{global max} = 32}$ (at $(4,0)$) $\boxed{\text{global min} = -\frac{9}{2} = -4.5}$ at $(\frac{3}{2}, 3)$

(c) You do not need to explain your work on this part.

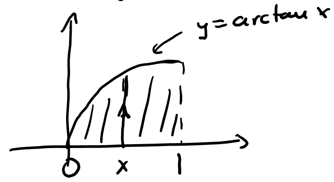
Does f have a global minimum on $\mathbf{R}^2$? Yes No

Does f have a global maximum on $\mathbf{R}^2$? Yes No

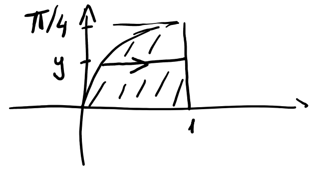
4. *(6 points per part)* Evaluate the following double integrals:

(a) $\displaystyle\int_0^1 \int_0^{\arctan(x)} \frac{1}{1-\tan(y)}\,dy\,dx$

Domain region: $0 \le x \le 1$
$0 \le y \le \arctan x$



Reversing: $0 \le y \le \pi/4$ & $\tan y \le x \le 1$



$$= \int_0^{\pi/4}\int_{\tan y}^{1} \frac{1}{1-\tan y}\,dx\,dy$$

$$= \int_0^{\pi/4} \frac{1}{1-\tan y}\left(x\Big|_{\tan y}^{1}\right)dy$$

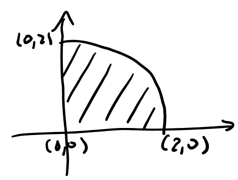
$$= \int_0^{\pi/4} \frac{1}{\cancel{1-\tan y}}\left(\cancel{1-\tan y}\right)dy$$

$$= y\Big|_0^{\pi/4} = \boxed{\frac{\pi}{4}}$$

(b) $\displaystyle\iint_D \cos(\sqrt{x^2+y^2})\,dA,$

where D is the region in the xy-plane given by $x^2+y^2 \le 4$, $x \ge 0$, and $y \ge 0$.

$$= \int_0^{\pi/2}\int_0^2 \cos(r)\, r\,dr\,d\theta \quad \text{(in polar coordinates!)}$$



$$= \underbrace{\left(\int_0^{\pi/2} d\theta\right)}\ \underbrace{\left(\int_0^2 r\cos r\,dr\right)}$$

Integration by Parts
$u = r \quad dv = \cos r\,dr$
$du = dr \quad v = \sin r$

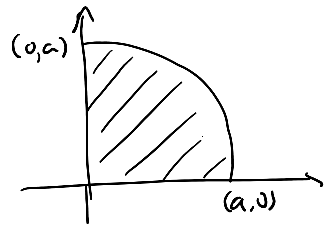
$$= \frac{\pi}{2}\left[r\sin r\Big|_0^2 - \int_0^2 \sin r\,dr\right]$$

$$= \frac{\pi}{2}\left[2\sin 2 + \cos r\Big|_0^2\right]$$

$$= \boxed{\frac{\pi}{2}\left[2\sin 2 + \cos 2 - 1\right]} \quad (\cong 0.63)$$

5. *(10 points)* In this problem, a denotes a positive constant. A flat plate occupies the part of the disk $x^2 + y^2 \le a^2$ that lies in the first quadrant. The plate has density $\rho(x, y) = xy^2$. The y-coordinate of the center of mass of the plate is $\bar{y} = 1$.

   Find a.



The mass of the plate is:

$$m = \iint_D \rho(x,y)\, dA = \iint_D xy^2\, dA$$

$$= \int_0^{\pi/2}\int_0^{a} r\cos\theta\,(r\sin\theta)^2\, r\, dr\, d\theta$$

$$= \int_0^{\pi/2}\int_0^{a} r^4 \cos\theta \sin^2\theta\, dr\, d\theta = \underbrace{\left(\int_0^{\pi/2}\cos\theta\,(\sin\theta)^2\, d\theta\right)}_{u = \sin\theta,\ du = \cos\theta\, d\theta}\left(\int_0^{a} r^4\, d\theta\right)$$

$$= \left(\int_0^1 u^2\, du\right)\left(\tfrac{1}{5}r^5\right)\Big|_{r=0}^{r=a} = \left(\tfrac{1}{3}u^3\Big|_0^1\right)\left(\tfrac{1}{5}a^5\right)$$

$$= \frac{1}{15}a^5$$

The y-coordinate is: $\bar{y} = \dfrac{1}{m}\iint_D y\,\rho(x,y)\, dA = \dfrac{15}{a^5}\iint_D xy^3\, dA$

In polar coordinates:

$$\bar{y} = \frac{15}{a^5}\int_0^{\pi/2}\int_0^{a}(r\cos\theta)(r\sin\theta)^3\, r\, dr\, d\theta$$

$$= \frac{15}{a^5}\left(\int_0^{\pi/2}\underbrace{(\sin\theta)^3}_{u^3}\,\underbrace{\cos\theta\, d\theta}_{du}\right)\left(\int_0^{a} r^5\, dr\right)$$

$$= \frac{15}{a^5}\left(\int_0^1 u^3\, du\right)\left(\int_0^{a} r^5\, dr\right) = \frac{15}{a^5}\left(\tfrac{1}{4}u^4\Big|_0^1\right)\left(\tfrac{1}{6}r^6\right)\Big|_0^{a}$$

$$= \frac{\cancel{15}^{\,5}}{a^5}\left(\frac{1}{4}\right)\left(\frac{a^6}{\cancel{6}_{\,2}}\right) = \frac{5a}{8}$$

We need: $\bar{y} = 1$ so $\dfrac{5a}{8} = 1$

$\therefore \boxed{a = 8/5}$.

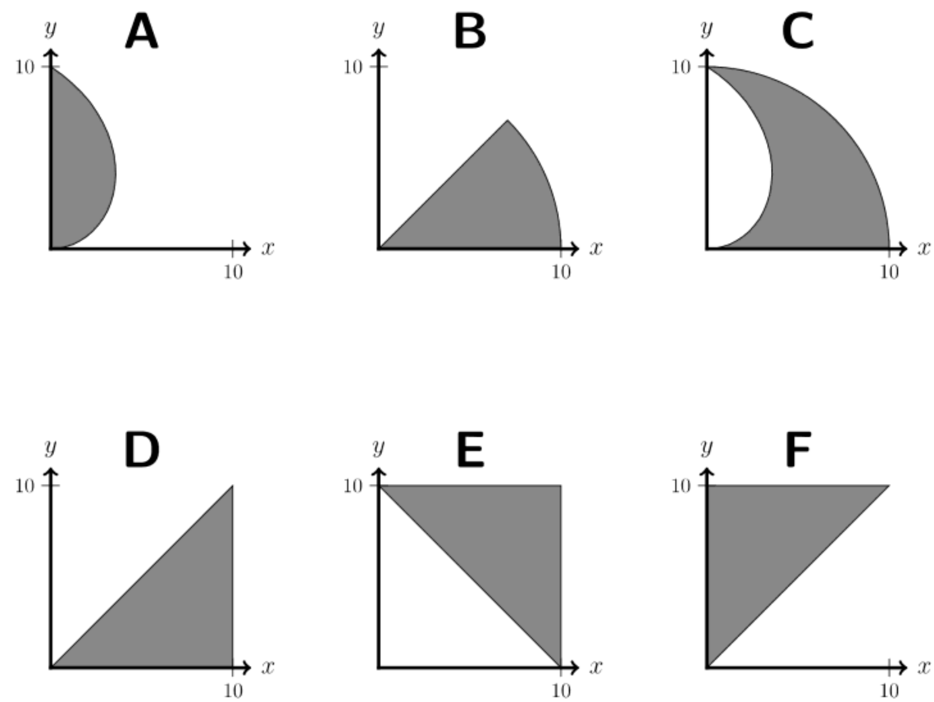
6. *(2 points per part)* Match each of the following area integrals with one of the regions (A)-(F) below, corresponding to its domain of integration. You do not need to justify. (Do NOT compute the integrals)

(a) $\int_0^{10} \int_0^{\pi/4} r \, d\theta \, dr$ matches region: B

(b) $\int_0^{10} \int_0^{y} dx \, dy$ matches region: F

(c) $\int_0^{\pi/2} \int_{\left(\frac{20}{\pi}\right)\theta}^{10} r \, dr \, d\theta$ matches region: C

(d) $\int_0^{10} \int_x^{10} dy \, dx$ matches region: F



7. (*7 points per part*) For both part (a) and (b), let $f(x) = \sqrt{(x-1)^5} - \cos(x-2)$.

(a) Find the second Taylor polynomial, $T_2(x)$, for $f(x)$ based at $b = 2$.

$f(x) = \sqrt{(x-1)^5} - \cos(x-2)$ $\quad f(2) = 0$

$f'(x) = \frac{5}{2}\sqrt{(x-1)^3} + \sin(x-2)$ $\quad f'(2) = \frac{5}{2}$

$f''(x) = \frac{15}{4}\sqrt{x-1} + \cos(x-2)$ $\quad f''(2) = \frac{19}{4}$

$$T_2(x) = \boxed{\frac{5}{2}(x-2) + \frac{19}{8}(x-2)^2}$$

(b) Use Taylor's inequality to find an upper bound (as sharp as possible) for $|f(x) - T_2(x)|$ on the interval $[1.8, 2.2]$.

$$f'''(x) = \underbrace{\frac{15}{8\sqrt{x-1}}}_{\text{greatest at } x=1.8} - \underbrace{\sin(x-2)}_{\text{lowest at } x=1.8}$$

on $[1.8, 2.2]$:

So $M = f'''(1.8) = \frac{15}{8\sqrt{.8}} - \sin(-0.2)$

≈ 2.295

So $|T_2(x) - f(x)| \le \frac{1}{6}(2.295)|0.2|^3 \approx \boxed{0.00306}$

8. *(14 points)* For this problem, you may use the following basic Taylor series:

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k, \quad e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad \sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}, \quad \cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

(a) Find the Taylor series for $f(x) = \dfrac{x}{5+x^2}$ based at $b = 0$. Express your answer using $\sum$–notation.

$$\frac{x}{5+x^2} = \frac{x}{5}\left(\frac{1}{1-\left(-\frac{x^2}{5}\right)}\right) = \frac{x}{5}\sum_{k=0}^{\infty}\left(-\frac{x^2}{5}\right)^k$$

1

$$= \frac{x}{5}\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{5^k}$$

$$= \boxed{\sum_{k=0}^{\infty}\frac{(-1)^k}{5^{k+1}}x^{2k+1}}$$

(b) Find the open interval of convergence for the series you found in (a).

$$\left|\frac{x^2}{5}\right| < 1 \iff x^2 < 5$$

$$\iff \boxed{-\sqrt{5} < x < \sqrt{5}}$$

(c) Use the Taylor series you found in (a) to find the Taylor series for $g(x) = \ln(5 + x^2)$ based at $b = 0$.

$$g'(x) = \frac{2x}{5+x^2} = \sum_{k=0}^{\infty}\frac{2(-1)^k}{5^{k+1}}x^{2k+1} \quad \text{by (a)}$$

$$\therefore g(x) = \int\left(\sum_{k=0}^{\infty}\frac{2(-1)^k}{5^{k+1}}x^{2k+1}\right)dx = \sum_{k=0}^{\infty}\frac{\cancel{2}(-1)^k}{5^{k+1}}\frac{x^{2k+2}}{\cancel{2k+2}_{\,k+1}}$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)5^{k+1}}x^{2k+2} + C = \frac{1}{5}x^2 - \frac{1}{10}x^4 + \frac{1}{75}x^6 + \cdots + C$$

$$g(0) = \ln 5 = C$$

$$\boxed{g(x) = \ln 5 + \sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)5^{k+1}}x^{2k+2}}$$

9. *(10 points)* Let $f(x, y) = 100 - x^2 - y^2 + xy$.

(a) Find a normal vector to the tangent plane to the surface $z = f(x, y)$ at the point $(x_0, y_0, f(x_0, y_0))$. (Your answer will depend on x_0 and y_0.)

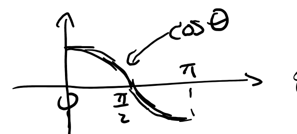
$f_x = -2x + y$, $f_y = -2y + x$

$$\boxed{\vec{n} = \langle -2x_0 + y_0,\ -2y_0 + x_0,\ -1 \rangle}$$

(b) Find a point P on the portion of the graph of f above the region $-2 \le x \le 2$, $-2 \le y \le 2$ where the tangent plane at P intersects the xy-plane at the largest possible (acute) angle.

The $\measuredangle$ of intersection between the tangent plane at P and the xy-plane = the $\measuredangle\ \theta$ between $\vec{n} = \langle -2x_0 + y_0, -2y_0 + x_0, -1 \rangle$ & $-\vec{k} = \langle 0, 0, -1 \rangle$

$$\cos\theta = \frac{\vec{n} \cdot (-\vec{k})}{|\vec{n}||\vec{k}|} = \frac{1}{|\vec{n}|}$$



Largest possible (acute) $\measuredangle\ \theta$ means smallest possible (positive) $\cos\theta$ which, in turn, mean largest possible $|\vec{n}|$, or, simpler, $|\vec{n}|^2$

So: we need to find the point $P = (x_0, y_0, z_0)$ in the square $-2 \le x, y \le 2$ that maximizes $N(x,y) = |\vec{n}|^2 = (-2x + y)^2 + (-2y + x)^2 + 1$

$$N(x,y) = 4x^2 - 4xy + y^2 + 4y^2 - 4xy + x^2 + 1 = 5x^2 + 5y^2 - 8xy + 1$$

① CP's: $\begin{cases} N_x = 10x - 8y = 0 \Rightarrow y = \frac{10x}{8} = \frac{5x}{4} \\ N_y = 10y - 8x = 0 \Rightarrow 10\left(\frac{5x}{4}\right) - 8x = 0 \Rightarrow x = 0 \Rightarrow y = 0 \end{cases}$ $\boxed{\text{only } (0,0),\ N(0,0) = 1}$

② Boundary: (i) $x = 2$: $g_1(y) = 20 + 5y^2 - 16y + 1 = 5y^2 - 16y + 21$ $\boxed{(2, 1.6),\ N(2, 1.6) = g_1(1.6) = 8.2}$

$g_1'(y) = 10y - 16 \Rightarrow y = 1.6$

(ii) $x = -2$: $g_2(y) = 20 + 5y^2 + 16y + 1 = 5y^2 + 16y + 21$

$\Rightarrow$ CP $(-2, -1.6)$ w/ $z(-2, -1.6) = g_2(-1.6) = 8.2$

(iii) & (iv) $y = \pm 2$: by symmetry: $\boxed{(1.6, 2)\ \&\ (-1.6, -2) \text{ w/ } z = 8.2}$

Corners: $\boxed{N(2,2) = 20 + 20 - 8(2)(2) + 1 = 9 = N(-2,-2)}$

$\boxed{N(-2,2) = 20 + 20 + 8(2)(2) + 1 = 73 = N(2,-2)}$ ← MAX

Maximum at $P = (-2, 2, f(2,-2)) = \boxed{(-2, 2, 88) \text{ OR } (2, -2, 88)}$ ← **ANSWER**