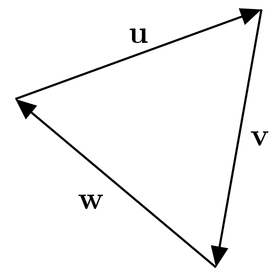
1. (8 points) Let **u**, **v**, and **w** be three-dimensional head-to-tail **unit** vectors in the xy-plane, forming an equilateral triangle, as pictured. Compute the following:

   (There is enough information provided to compute a specific value or vector as an answer to each of these questions.)



   (a) $\mathbf{u} \cdot \mathbf{u} =$ $|\vec{u}|^2 = \boxed{1}$

   (b) $\mathbf{u} \cdot \mathbf{v} =$ $|\vec{u}|\,|\vec{v}| \cos\theta = 1 \cdot 1 \cdot \frac{-1}{2} = \boxed{\frac{-1}{2}}$

   (c) $\mathbf{u} + \mathbf{v} + \mathbf{w} =$ $\boxed{\vec{0}}$ (since they form a cycle)

   (d) $|\mathbf{u} \times (\mathbf{v} \times \mathbf{w})| =$

   $|\vec{v} \times \vec{w}| = 1 \cdot 1 \cdot \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$

   $|\vec{u} \times (\vec{v} \times \vec{w})| = 1 \cdot \frac{\sqrt{3}}{2} \cdot \sin\left(\frac{\pi}{2}\right) = \boxed{\frac{\sqrt{3}}{2}}$

2. (6 points) Answer the following (unrelated) questions. No need to justify your answer.

   (a) Give an example of the equation of a plane that is perpendicular to the plane $z = x - y$. Write your answer in the form $Ax + By + Cz = D$.

   normal $\langle 1, -1, -1 \rangle$

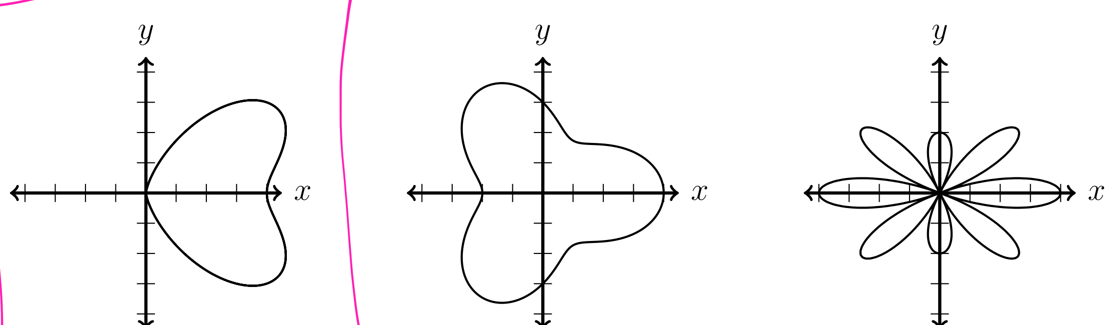
   $\boxed{x + y = 0}$, for example (many answers ok)

   (b) Which of the following is a level curve (i.e. a trace when $z = k$, a constant) for the surface $x^2 - y^2 + z = 0$? Circle all that apply.

   ELLIPSE / PARABOLA / HYPERBOLA / PAIR OF LINES

   ($z = 0$)

   (c) One of these is the graph of $r = 5\cos(\theta) - \cos(5\theta)$. Circle it.



3. (12 points) Consider the helix-like curve $\mathbf{r}(t) = \left\langle \cos(t), \sin(t), \frac{2t^{3/2}}{3} \right\rangle,\ t \geq 0.$

(a) Write a Cartesian equation for a surface that contains this curve.

$$\boxed{x^2 + y^2 = 1} \qquad \left(\text{because } \cos^2 t + \sin^2 t = 1\right)$$

(b) Compute the arc length of this curve from $t = 0$ to $t = a$, where a is an arbitrary positive constant.

$\vec{r}\,'(t) = \langle -\sin t, \cos t, \sqrt{t} \rangle$

$$\int_0^a \sqrt{\sin^2 t + \cos^2 t + (\sqrt{t})^2}\, dt = \int_0^a \sqrt{1+t}\, dt = \int_1^{a+1} \sqrt{u}\, du = \left(\frac{2u^{3/2}}{3}\right)\Bigg]_1^{a+1} = \boxed{\frac{2}{3}\left((a+1)^{3/2} - 1\right)}$$

$u = 1+t$
$du = dt$

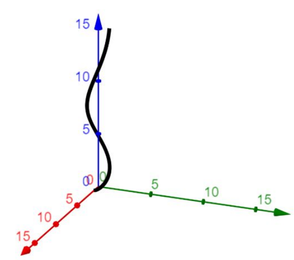
(c) Compute the curvature $\kappa(t)$ of this curve, and determine $\lim_{t\to\infty} \kappa(t)$.

$\vec{r}\,'(t) = \langle -\sin t, \cos t, \sqrt{t} \rangle \rightarrow |\vec{r}\,'(t)| = \sqrt{1+t}$

$\vec{r}\,''(t) = \left\langle -\cos t, -\sin t, \frac{1}{2\sqrt{t}} \right\rangle$

$\vec{r}\,'(t) \times \vec{r}\,''(t) = \left\langle \frac{\cos(t)}{2\sqrt{t}} + \sin(t)\sqrt{t}, -\sqrt{t}\cos(t) + \frac{\sin(t)}{2\sqrt{t}}, 1 \right\rangle$

$|\vec{r}\,'(t) \times \vec{r}\,''(t)| = \sqrt{\frac{1}{4t} + t + 1}$



$$\boxed{K(t) = \frac{\sqrt{\frac{1}{4t} + t + 1}}{\left(\sqrt{t+1}\right)^3}}$$

grows much faster than num.

$$\boxed{\lim_{t\to\infty} K(t) = 0}$$

4. (12 points) Find and classify all critical points of the function $f(x,y) = 4x^3 - 2x + 2xy - y^2$.

$f_x(x,y) = 12x^2 - 2 + 2y = 0 \rightarrow 12x^2 + 2x - 2 = 0$

$f_y(x,y) = 2x - 2y = 0 \rightarrow x = y$

$2(3x-1)(2x+1) = 0$

$\left(\frac{1}{3}, \frac{1}{3}\right)$ and $\left(\frac{-1}{2}, \frac{-1}{2}\right)$

To classify:

$f_{xx}(x,y) = 24x$

$f_{yy}(x,y) = -2$

$f_{xy}(x,y) = 2$

$D\left(\frac{1}{3}, \frac{1}{3}\right) = 8(-2) - 2^2 < 0$

$D\left(\frac{-1}{2}, \frac{-1}{2}\right) = (-12)(-2) - 2^2 > 0$

Saddle point @ $\left(\frac{1}{3}, \frac{1}{3}\right)$
Local max @ $\left(\frac{-1}{2}, \frac{-1}{2}\right)$

5. (12 points) Let $z = f(x, y)$ be the function defined by the implicit equation

$$z\sqrt{x^2 + y^2} + e^{z+1} + y = 0.$$

(a) Compute $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ at the point $(3, 4, -1)$.

$\dfrac{\partial}{\partial x}$: $\quad \dfrac{\partial z}{\partial x}\sqrt{x^2+y^2} + \dfrac{xz}{\sqrt{x^2+y^2}} + \dfrac{\partial z}{\partial x}e^{z+1} = 0$

$$5\frac{\partial z}{\partial x} - \frac{3}{5} + \frac{\partial z}{\partial x} = 0 \qquad \boxed{\frac{\partial z}{\partial x} = \frac{1}{10}}$$

$\dfrac{\partial}{\partial y}$: $\quad \dfrac{\partial z}{\partial y}\sqrt{x^2+y^2} + \dfrac{yz}{\sqrt{x^2+y^2}} + \dfrac{\partial z}{\partial y}e^{z+1} + 1 = 0$

$$5\frac{\partial z}{\partial y} - \frac{4}{5} + \frac{\partial z}{\partial y} + 1 = 0 \qquad \boxed{\frac{\partial z}{\partial y} = \frac{-1}{30}}$$

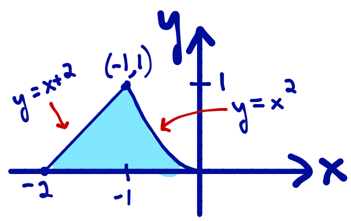
(b) Use linear approximation to estimate the value of $f(3.07, 4.12)$.

$$f(3.07, 4.12) \approx \frac{1}{10}(3.07-3) - \frac{1}{30}(4.12-4) + (-1) = \boxed{-0.997}$$

6. (13 points) Suppose $f(x, y)$ is continuous and D is a region in the xy-plane such that

$$\iint_D f(x,y)\,dA = \int_0^1 \int_{y-2}^{-\sqrt{y}} f(x,y)\,dx\,dy.$$

(a) Sketch D and reverse the order of integration.



$$\boxed{\int_{-2}^{-1}\int_0^{x+2} f(x,y)\,dy\,dx + \int_{-1}^{0}\int_0^{x^2} f(x,y)\,dy\,dx}$$

(b) Let D be the region described above and suppose a lamina in the shape of D has variable density $\rho(x, y) = -12x$. Compute the mass of the lamina.

(You do not need to use the same setup from part (a) to compute this.)

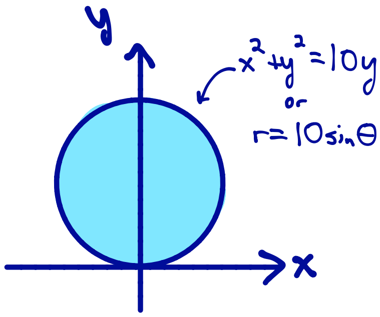
$$\int_0^1 \int_{y-2}^{-\sqrt{y}} -12x\,dx\,dy = \int_0^1 \left(-6x^2\right)\Big]_{y-2}^{-\sqrt{y}} dy = \int_0^1 \left(-6y + 6(y-2)^2\right) dy$$

$$= \int_0^1 \left(6y^2 - 30y + 24\right) dy = \left(2y^3 - 15y^2 + 24y\right)\Big]_0^1 = \boxed{11}$$

7. (12 points) Use polar coordinates to find the volume of the solid below the cone $\sqrt{x^2+y^2} = 5z$, above the xy-plane, and inside the cylinder $x^2+y^2 = 10y$.

$$x^2 + (y-5)^2 = 25$$

$$z = \frac{\sqrt{x^2+y^2}}{5}$$



$$\int_0^{\pi}\int_0^{10\sin\theta} \left(\frac{r}{5}\right) r\,dr\,d\theta$$

$$= \int_0^{\pi} \left(\frac{r^3}{15}\right)\Big]_0^{10\sin\theta} d\theta$$

$$= \int_0^{\pi} \frac{1000}{15}\sin^3\theta\,d\theta$$

$$= \frac{200}{3}\int_0^{\pi} \sin\theta\left(1-\cos^2\theta\right)d\theta = \frac{200}{3}\int_1^{-1} -\left(1-u^2\right)du$$

$u = \cos\theta$
$du = -\sin\theta\,d\theta$

$$= \frac{200}{3}\left(-u+\frac{u^3}{3}\right)\Big]_1^{-1}$$

$$= \frac{200}{3}\left(\left(1-\frac{1}{3}\right)-\left(-1+\frac{1}{3}\right)\right)$$

$$= \boxed{\frac{800}{9}}$$

8. (13 pts) For **ALL** parts below, let $f(x) = \dfrac{4x}{2x+1} - xe^{6x}$ and $b = 0$.

(a) Give the Taylor series for $f(x)$ based at b.
Give your final answer in Sigma notation using one Sigma sign.

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \qquad e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$\frac{1}{2x+1} = \frac{1}{1-(-2x)} = \sum_{k=0}^{\infty} (-2x)^k = (-2)^k x^k \qquad e^{6x} = \sum_{k=0}^{\infty} \frac{(6x)^k}{k!} = \sum_{k=0}^{\infty} \frac{6^k x^k}{k!}$$

$$\frac{4x}{2x+1} = \sum_{k=0}^{\infty} 4(-2)^k x^{k+1} \qquad xe^{6x} = \sum_{k=0}^{\infty} \frac{6^k x^{k+1}}{k!}$$

$$\frac{4x}{2x+1} - xe^{6x} = \boxed{\sum_{k=0}^{\infty} \left(4(-2)^k - \frac{6^k}{k!}\right) x^{k+1}}$$

(b) Give the open interval of convergence for your answer in part (a).

$\frac{1}{1-x}$ converges when $-1 < x < 1$.

$\frac{1}{1-(-2x)}$ converges when $-1 < -2x < 1$: $\boxed{\left(\frac{-1}{2}, \frac{1}{2}\right)}$

Other operations don't change this.

(c) Use the first three nonzero terms of the Taylor series to estimate the value of $f\left(\frac{1}{10}\right)$.
Give your final answer to three digits after the decimal.

$$\sum_{k=0}^{\infty} \left(4(-2)^k - \frac{6^k}{k!}\right) x^{k+1} = 3x - 14x^2 - 2x^3 + \cdots$$

$$f\left(\frac{1}{10}\right) \approx \frac{3}{10} - \frac{14}{100} - \frac{2}{1000} = \boxed{.158}$$

9. (12 pts) For **ALL** parts below, consider Taylor polynomials for $g(x) = e^{x/2}$ based at $b = 1$.

(a) Find the third Taylor polynomial, $T_3(x)$, for $g(x)$ based at $b = 1$.

$f(x) = e^{x/2}$ $\quad f(b) = \sqrt{e}$

$f'(x) = \frac{1}{2} e^{x/2}$ $\quad f'(b) = \frac{1}{2}\sqrt{e}$

$f''(x) = \frac{1}{4} e^{x/2}$ $\quad f''(b) = \frac{1}{4}\sqrt{e}$

$f'''(x) = \frac{1}{8} e^{x/2}$ $\quad f'''(b) = \frac{1}{8}\sqrt{e}$

$$\boxed{T_3(x) = \sqrt{e} + \frac{1}{2}\sqrt{e}(x-1) + \frac{1}{8}\sqrt{e}(x-1)^2 + \frac{1}{48}\sqrt{e}(x-1)^3}$$

(b) On the interval $I = [0, 2]$, for which of the values of n below does Taylor's inequality guarantee that $|f(x) - T_n(x)| < 0.001$?
You **must** show enough error bound calculations to justify your answer.

Circle **ALL** that apply: $\quad n = 2 \quad n = 3 \quad n = 4 \quad n = 5 \quad n = 6$

$f^{(n)}(x) = \frac{1}{2^n} e^{x/2}$

On $[0,2]$ this is at most $\frac{1}{2^n} e$.

So: $\left|f(x) - T_n(x)\right| < \left(\frac{1}{(n+1)!}\right)\underbrace{\left(\frac{1}{2^{n+1}} e\right)}_{M} 1^{n+1} = \frac{e}{2^{n+1}(n+1)!}$

$n=2$: $\frac{e}{2^3 \cdot 3!} \approx .057$ no.

$n=3$: $\frac{e}{2^4 \cdot 4!} \approx .007$ no.

$n=4$: $\frac{e}{2^5 \cdot 5!} \approx .0007$ yeah!