Math 126 Final Examination Spring 2005

Your Name

Your Signature

Student ID # Quiz Section

Professor’s Name TA’s Name

• This exam is closed book. You may use one $8\frac{1}{2} \times 11$ sheet of notes.

• Give your answers in exact form. Do not give decimal approximations.

• Graphing calculators are not allowed. Do not share notes.

• In order to receive credit, you must show your work. Do not do computations in your head. Instead, write them out on the exam paper.

• Place a box around **YOUR FINAL ANSWER** to each question.

• If you need more room, use the backs of the pages and indicate to the reader that you have done so.

• Raise your hand if you have a question.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Total Points</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Problem</th>
<th>Total Points</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>14</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>100</td>
<td></td>
</tr>
</tbody>
</table>
1. [12 points total] Consider the function $f(x) = \sin \left( \frac{\pi x}{6} \right)$.

(a) [6 points] Find $T_2(x)$, the second order Taylor polynomial for $f(x)$ centered at $a = 1$.

(b) [6 points] Use Taylor’s Inequality to find an upper bound on $|f(1.1) - T_2(1.1)|$. 
2. [6 points total] Determine whether or not the infinite series \( \sum_{n=1}^{\infty} (-1)^n n e^{-n} \) converges. Justify your answer.
3. [10 points total] Let \( L_1 \) be the line given by the parametric equations
\[
x = 2t, \quad y = 0, \quad z = 4 - 4t,
\]
and let \( L_2 \) be the line given by the parametric equations
\[
x = 2 - 2u, \quad y = 3u, \quad z = 0.
\]

(a) [4 points] Find the point of intersection of \( L_1 \) and \( L_2 \).

(b) [6 points] Find an equation of the plane that contains both \( L_1 \) and \( L_2 \). Give your answer in the form \( ax + by + cz = d \).
4. [8 points] The curved portion of a skateboard ramp has a cross section given by the inverted cycloid (for maximal speed at the bottom):

\[
\begin{align*}
  x &= r(t - \sin(t)) \\
  y &= r(1 + \cos(t))
\end{align*}
\]

for \(0 \leq t \leq \pi\). Find \(r\) so that the length of the curved portion is 8 feet. (Hint: To compute the integral, you might find the trig identity \(1 - \cos \theta = 2 \sin^2(\theta/2)\) useful.)
5. [12 points total] You are assigned the task of building the electronics for an antenna on a spacecraft so that the antenna will always point toward a space station. On board are accelerometers which measure the acceleration in the $x$, $y$ and $z$ direction. Choose a coordinate system so that the space station is at the origin $(0, 0, 0)$, and assume that the spacecraft starts at the space station with initial velocity $\langle 0, 0, 0 \rangle$. The acceleration at time $t$ is given by

$$a(t) = \langle e^t, \cos(t), t \rangle.$$ 

(a) [4 points] Find the velocity as a vector function of time.

(b) [4 points] Find the position as a vector function of time.

(c) [4 points] Find a unit vector (which depends on time) that points from the spacecraft to the space station at time $t$. DO NOT SIMPLIFY YOUR ANSWER.
6. [6 points total] Using the fact that $\frac{d}{du}(\tan^{-1} u) = \frac{1}{1+u^2}$, find the Maclaurin series for $g(x) = \tan^{-1} \left[(x/2)^3\right]$. 
7. [6 points total] Find a vector \( \mathbf{u} \) which satisfies both of the following conditions:

(i) \( \mathbf{u} \) is orthogonal to \( \langle 2, 1, 4 \rangle \),

(ii) the cross product of \( \mathbf{u} \) and \( \langle 1, 2, 0 \rangle \) equals \( \langle 2, -1, 0 \rangle \)
8. [10 points total] Let $C$ be the curve defined by the vector function

$$\mathbf{r}(t) = \langle e^t \sin t, \ e^t \cos t, \ e^{-2t} \rangle.$$ 

(a) [4 points] Find a vector equation of the tangent line at $t = 0$. (Be sure your answer is an equation.)

(b) [6 points] Find the curvature $\kappa$ at $t = 0$. 


9. [14 points total] Let \( f(x, y) = \ln \left(x^2 + \sqrt{y}\right) + e^y \cos x. \)

(a) [3 points] Find the partial derivative \( f_x. \)

(b) [3 points] Find the partial derivative \( f_y. \)

(c) [4 points] Find the second order partial derivative \( f_{xy}. \)

(d) [4 points] Find an equation of the tangent plane to the graph of \( z = f(x, y) \) at the point \((0, 1, e).\)
10. [8 points total] Evaluate the iterated integral \[ \int_0^1 \int_{x^2}^1 x \sin(\pi y^2) \, dy \, dx. \]
11. [8 points total] Evaluate the double integral \( \iint_D x^2 + x + y^2 \, dA \), where \( D \) is the region

\[
D = \{(x, y) : x^2 + y^2 \leq 4 \text{ and } y \geq x\}.
\]