Name ________________________ Quiz Section ______

This worksheet helps you review improper intergrals and approximation techniques.

Improper Integrals

1 Show that $\int_1^\infty \frac{1-\sin x}{x^2}\,dx$ converges, using the Comparison Test.

We know that $0 \leq 1-\sin(x) \leq 2$ *so* $0 \leq \frac{1-\sin(x)}{x^2} \leq \frac{2}{x^2}$.

Since $\int_1^\infty \frac{2}{x^2}\,dx = 2\int_1^\infty \frac{1}{x^p}\,dx$ *with* $p = 2 > 1$, *we know that it is convergent.*

Thus $\int_1^\infty \frac{1-\sin x}{x^2}\,dx$ *converges by the Comparison Test.*

2 Use Problem 1 and the fact that $\int_1^\infty \frac{1}{x^2}\,dx$ converges to show that $\int_1^\infty \frac{\sin x}{x^2}\,dx$ converges. Why can't we use the Comparison Test directly on this one?

Write $\int_1^\infty \frac{\sin x}{x^2}\,dx = \int_1^\infty \frac{1}{x^2}\,dx - \int_1^\infty \frac{1-\sin x}{x^2}\,dx$.

Since it is the difference of two convergent integrals, it must converge.

We cannot use the Comparison Test directly because it only applies to positive valued functions.

3 Use Problem 2 and integration-by-parts to show that $\int_1^\infty \frac{\cos x}{x}\,dx$ converges.

Let $u = \frac{1}{x}$ *and* $dv = \cos(x)\,dx$. *Then*

$$\begin{aligned}\int_1^\infty \frac{\cos x}{x}\,dx &= \lim_{b\to\infty} \frac{\sin(x)}{x}\Big|_1^b - \int_1^\infty \frac{\sin x}{x^2}\,dx \\ &= -\sin(1) - \int_1^\infty \frac{\sin x}{x^2}\,dx\end{aligned}$$

So the integral converges by Problem 2.

Approximation Techniques

Let $f(x) = \dfrac{\ln(x)}{x}$. We will use geometric reasoning to see if the Trapezoid Rule gives an overestimate, or an underestimate of the integral $\int_1^3 f(x)\,dx$.

1 Compute $f'(x)$. Where is the function increasing and decreasing? (We only care about the interval $1 \leq x \leq 3$.)

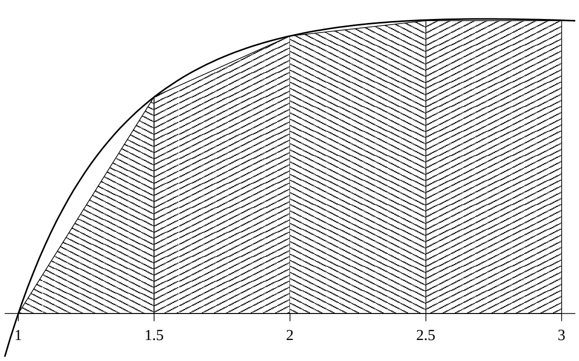
$f'(x) = \dfrac{1 - \ln(x)}{x^2}$ *so* $f(x)$ *is increasing when* $1 \leq x < e$ *and decreasing when* $e < x \leq 3$.

2 Compute $f''(x)$. Is the function concave up or concave down on the interval?

$f''(x) = \dfrac{-3 + 2\ln(x)}{x^3}$ *so* $f(x)$ *is concave down for* $0 < x < e^{3/2} \approx 4.48$.

Thus $f''(x)$ *is concave down on the whole interval* $1 \leq x \leq 3$.

3 Sketch a graph of $y = f(x)$ on the interval $1 \leq x \leq 3$. Sketch the approximate area under the curve given by the Trapezoid Rule with $n = 4$ subintervals. Is it an overestimate or an underestimate? How is the answer related to Problem 2?



You can see in the picture that it is an underestimate. The point is that, when a curve is concave down, any secant line will lie below it. Thus the Trapezoid Rule will give an underestimate.