The following integrals are more challenging than the basic ones we've seen in the textbook so far. You will probably have to use more than one technique to solve them. Don't hesitate to ask for hints if you get stuck.

1. $\displaystyle\int \frac{\sin(t)\cos(t)}{\sin^2(t)+6\sin(t)+8}\,dt$

Let $x=\sin(t)$*. Then* $dx=\cos(t)\,dt$ *and the integral transforms to* $\displaystyle\int \frac{x}{x^2+6x+8}\,dx.$

Now use Partial Fractions: $\displaystyle\frac{x}{x^2+6x+8}=\frac{x}{(x+2)(x+4)}=\frac{A}{x+2}+\frac{B}{x+4}.$

Multiply both sides by x^2+6x+8 *to get* $x=A(x+4)+B(x+2)$*.*

If $x=-2$*, we get* $-2=2A$ *so* $A=-1$*.*

If $x=-4$*, we get* $-4=-2B$ *so* $B=2$*.*

Thus we need to compute $\displaystyle\int \frac{2}{x+4}-\frac{1}{x+2}\,dx=2\ln|x+4|-\ln|x+2|+C=\ln\left|\frac{(x+4)^2}{x+2}\right|+C$

Now reverse the substitution $x=\sin(t)$ *to get the answer* $\displaystyle\ln\left(\frac{[\sin(t)+4]^2}{\sin(t)+2}\right)+C$

(Note that we can drop the absolute value signs, since the arguement to the logarithm is positive.)

2. $\displaystyle\int \left(\sin^{-1}(x)\right)^2\,dx$

Use Integration by Parts twice:

Let $u=\left(\sin^{-1}(x)\right)^2$ *and* $dv=dx$*. Then* $du=\dfrac{2\sin^{-1}(x)}{\sqrt{1-x^2}}\,dx$ *and* $v=x$*.*

$$\int \left(\sin^{-1}(x)\right)^2\,dx = x\left(\sin^{-1}(x)\right)^2-\int\frac{2x\sin^{-1}(x)}{\sqrt{1-x^2}}\,dx$$

Now let $U=2\sin^{-1}(x)$ *and* $dV=\dfrac{x}{\sqrt{1-x^2}}\,dx$*. Then* $dU=\dfrac{2}{\sqrt{1-x^2}}$ *and* $V=-\sqrt{1-x^2}$*.*

$$\int\frac{2x\sin^{-1}(x)}{\sqrt{1-x^2}}\,dx=-2\sqrt{1-x^2}\,\sin^{-1}(x)-\int -2\,dx=-2\sqrt{1-x^2}\,\sin^{-1}(x)+2x+C$$

So the answer is $\displaystyle x\left(\sin^{-1}(x)\right)^2+2\sqrt{1-x^2}\,\sin^{-1}(x)-2x+C$

3. $\displaystyle\int \frac{y^2}{(1-y^2)^{7/2}}\,dy$

I'm sure there's more than one way to do this. I first did Integration by Parts:

Let $u = y$ *and* $dv = \dfrac{y}{(1-y^2)^{7/2}}\,dy$. *Then* $du = dy$ *and* $v = \dfrac{1}{5(1-y^2)^{5/2}}$.

$$\int \frac{y^2}{(1-y^2)^{7/2}}\,dy = \frac{y}{5(1-y^2)^{5/2}} - \int \frac{1}{5(1-y^2)^{5/2}}\,dy.$$

Then I used the Inverse Trig Substitution $y = \sin(\theta)$. *This transforms* $\displaystyle\int \frac{1}{5(1-y^2)^{5/2}}\,dy$ *to*

$$\frac{1}{5}\int \sec^4(\theta)\,d\theta = \frac{1}{5}\int \left(1+\tan^2(\theta)\right)\sec^2(\theta)\,d\theta.$$

Now the substitution $x = \tan(\theta)$ *transforms the integral to*

$$\frac{1}{5}\int 1 + x^2\,dx = \frac{1}{5}x + \frac{1}{15}x^3 + C.$$

Reverse the substitution with $x = \tan(\theta) = \dfrac{y}{\sqrt{1-y^2}}$ *to get the second integral:*

$$\int \frac{1}{5(1-y^2)^{5/2}}\,dy = \frac{1}{5}\left(\frac{y}{\sqrt{1-y^2}}\right) + \frac{1}{15}\left(\frac{y}{\sqrt{1-y^2}}\right)^3 + C$$

and the final answer is $$\frac{y}{5(1-y^2)^{5/2}} - \frac{1}{5}\left(\frac{y}{\sqrt{1-y^2}}\right) - \frac{1}{15}\left(\frac{y}{\sqrt{1-y^2}}\right)^3 + C$$

4. $\displaystyle\int \frac{1}{x+2\sqrt{x}+1}\,dx$

First do a Rationalizing Substitution: Let $u^2 = x$ *so that* $2u\,du = dx$.

This transforms the integral to $\displaystyle\int \frac{2u}{u^2+2u+1}\,du = \int \frac{2u}{(u+1)^2}\,du.$

Now do the elementary substitution $v = u+1$ *and* $dv = du$ *to get*

$$\int \frac{2(v-1)}{v^2}\,dv = \int 2v^{-1} - 2v^{-2}\,dv = 2\ln|v| + \frac{2}{v} + C.$$

Reversing the substititions $v = 1+u = 1+\sqrt{x}$ *gives:* $$2\ln\left|1+\sqrt{x}\right| + \frac{2}{1+\sqrt{x}} + C$$