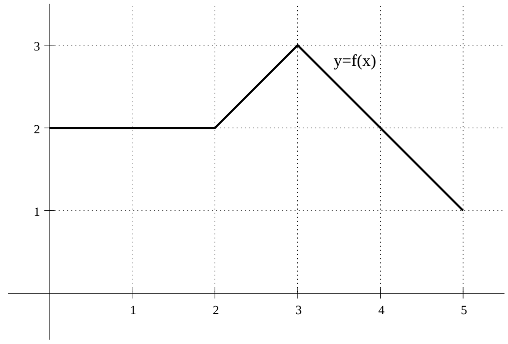
In this worksheet, we explore the Fundamental Theorem of Calculus and applications of the Area Problem to problems involving distance and velocity.

FTC Practice

1 Let $f(x)$ be given by the graph to the right and define $A(x) = \int_0^x f(t)\,dt$. Compute the following.



$A(1) =$ 2 $A(2) =$ 4

$A(3) =$ $6\frac{1}{2}$ $A(4) =$ 9

$A'(1) =$ 2 $A'(2) =$ 2

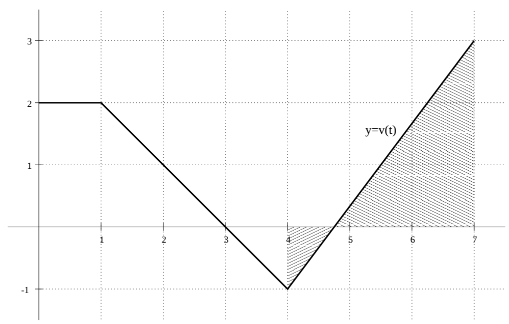
$A'(3) =$ 3 $A'(4) =$ 2

The maximum value of $A(x)$ on the interval $[0, 5]$ is $10\frac{1}{2}$

The maximum value of $A'(x)$ on the interval $[0, 5]$ is 3

Velocity and Distance

2 A toy car is travelling on a straight track. Its velocity $v(t)$, in m/sec, be given by the graph to the right. Define $s(t)$ to be the position of the car in meters. Choose coordinates so that $s(0) = 0$. Compute the following.



$s(2) =$ $3\frac{1}{2}$ $s(4) =$ $3\frac{1}{2}$ $s(6) =$ $4\frac{1}{6}$*

$v(2) =$ 1 $v(4) =$ -1 $v(6) =$ $1\frac{2}{3}$

The maximum value of $s(t)$ on the interval $[0, 7]$ is $6\frac{1}{2}$

The minimum value of $s(t)$ on the interval $[0, 7]$ is 0

The maximum value of $v(t)$ on the interval $[0, 7]$ is 3

The minimum value of $v(t)$ on the interval $[0, 7]$ is -1

* *This one's a little tricky.* $s(6) = s(4) - \left(\frac{1}{2} \times \frac{3}{4} \times 1\right) + \left(\frac{1}{2} \times \frac{5}{4} \times \frac{5}{3}\right)$.

Net and Total Change

3 (a) Evaluate $\int_{-2}^{2} \left|x^2 - 4\right| \, dx$ and $\left|\int_{-2}^{2} \left(x^2 - 4\right) \, dx\right|$ and explain your answers.

First find where $x^2 - 4$ *is positive and negative. Since* $x^2 - 4 = (x+2)(x-2)$,

$$x^2 - 4 > 0 \qquad \textit{for } x < -2 \textit{ and for } x > 2,$$

and

$$x^2 - 4 < 0 \qquad \textit{for } -2 < x < 2.$$

So $x^2 - 4$ *is negative on the whole interval* $(-2, 2)$.

$$\int_{-2}^{2} \left|x^2 - 4\right| \, dx = \int_{-2}^{2} -\left(x^2 - 4\right) \, dx = \left(-\frac{1}{3}x^3 + 4x\right)\Big|_{-2}^{2} = \frac{16}{3} + \frac{16}{3} = \frac{32}{3},$$

while

$$\left|\int_{-2}^{2} \left(x^2 - 4\right) \, dx\right| = \left|\left(\frac{1}{3}x^3 - 4x\right)\Big|_{-2}^{2}\right| = \left|-\frac{16}{3} - \frac{16}{3}\right| = \frac{32}{3}.$$

These are the same because $x^2 - 4$ *has the same sign (negative) throughout the interval* $-2 < x < 2$.

(b) Now evaluate $\int_{-3}^{3} \left|x^2 - 4\right| \, dx$ and $\left|\int_{-3}^{3} \left(x^2 - 4\right) \, dx\right|$ and explain your answers.

Since $x^2 - 4 > 0$ *for* $-3 < x < -2$, $x^2 - 4 < 0$ *for* $-2 < x < 2$, *and* $x^2 - 4 > 0$ *for* $2 < x < 3$,

$$\begin{aligned} \int_{-3}^{3} \left|x^2 - 4\right| \, dx &= \int_{-3}^{-2} \left(x^2 - 4\right) \, dx + \int_{-2}^{2} -\left(x^2 - 4\right) \, dx + \int_{2}^{3} \left(x^2 - 4\right) \, dx \\ &= \left(\frac{1}{3}x^3 - 4x\right)\Big|_{-3}^{-2} + \left(-\frac{1}{3}x^3 + 4x\right)\Big|_{-2}^{2} + \left(\frac{1}{3}x^3 - 4x\right)\Big|_{2}^{3} \\ &= \left(\frac{16}{3} - 3\right) + \left(\frac{16}{3} + \frac{16}{3}\right) + \left(-3 + \frac{16}{3}\right) = \frac{46}{3}, \end{aligned}$$

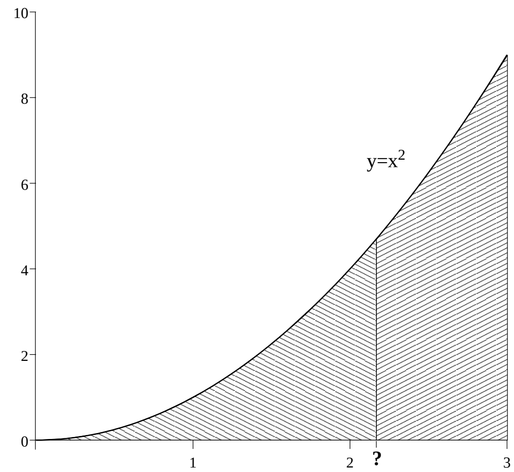
while

$$\left|\int_{-3}^{3} \left(x^2 - 4\right) \, dx\right| = \left|\left(\frac{1}{3}x^3 - 4x\right)\Big|_{-3}^{3}\right| = |-3 - 3| = 6.$$

These are not the same because $x^2 - 4$ *changes sign in the interval* $-3 < x < 3$.

Another Area Problem

4 An artist you know wants to make a figure consisting of the region between the curve $y = x^2$ and the x-axis for $0 \le x \le 3$



(i) Where should the artist divide the region with a vertical line so that each piece has the same area? (See the picture.)

(ii) Where should the artist divide the region with vertical lines to get 3 pieces with equal areas?

First compute $\int_0^3 x^2\,dx = 9$.

(i) Let a be the value we are looking for. Then we need $\int_0^a x^2\,dx = 9/2$. *This gives the equation* $\frac{1}{3}a^3 = 9/2$. *Solving, we get* $a = 3/\sqrt[3]{2} \approx 2.38$.

(ii) Let b and c be the values we are looking for. We need $\int_0^b x^2\,dx = 9/3 = 3$ *and* $\int_c^3 x^2\,dx = 3$. *The first integral gives* $\frac{1}{3}b^3 = 3$ *so* $b = \sqrt[3]{9} \approx 2.08$. *The second integral gives* $9 - \frac{1}{3}c^3 = 3$ *so* $c = \sqrt[3]{18} \approx 2.62$.