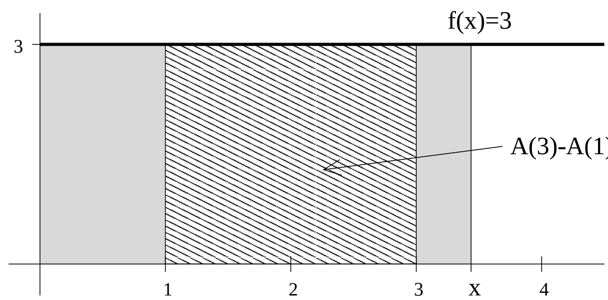
Introduction

Many interesting and useful functions can be defined as the area under some other function. There is a very nice relationship between the original function and the area function. We will explore that relationship in this worksheet.

Area Functions

1a Define $A(x)$ to be the **area** bounded by the x-axis and the function $f(x) = 3$ between the y-axis and the vertical line at x. (See the diagram.)



$A(1) =$ 3 $A(2) =$ 6

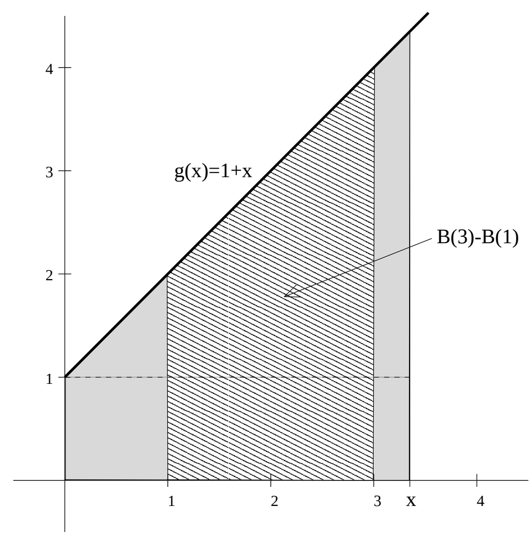
$A(3) =$ 9 $A(4) =$ 12

and, in general,

$A(x) =$ $3x$ (a formula)

Shade the region whose area is $A(3) - A(1)$.

1b Define $B(x)$ to be the **area** bounded by the x-axis and the function $g(x) = 1 + x$ between the y-axis and the vertical line at x. (See the diagram.)



$B(1) =$ 3/2 $B(2) =$ 4

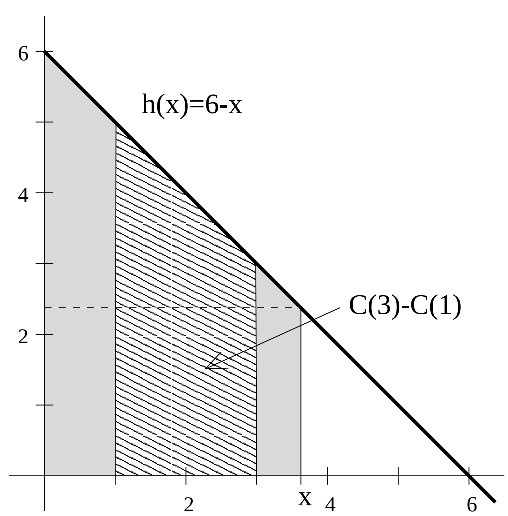
$B(3) =$ 15/2 $B(4) =$ 12

and, in general,

$B(x) =$ $x + \frac{1}{2}x^2$ (a formula)
(Hint: think triangle + rectangle)

Shade the region whose area is $B(3) - B(1)$.

1c Define $C(x)$ to be the **area** bounded by the x-axis and the function $h(x) = 6 - x$ between the y-axis and the vertical line at x. (See the diagram.)



$C(1) =$ 11/2 $C(2) =$ 10

$C(3) =$ 27/2 $C(4) =$ 16

and, in general,

$C(3) =$ $6x - \frac{1}{2}x^2$ (a formula)

Shade the region whose area is $C(3) - C(1)$.

For each of the above, the **area** increases as x increases. So $A(x)$, $B(x)$ and $C(x)$ are increasing functions even though $f(x)$ is constant, $g(x)$ is increasing and $h(x)$ is decreasing. (There is a difficulty with $C(x)$ when x gets larger than 6. We'll deal with that later.)

1d Now calculate the derivatives of the area functions from problems 1, 2 and 3 above:

$A'(x) =$ ___ 3 ___ $B'(x) =$ ___ $1+x$ ___ $C'(x) =$ ___ $6-x$ ___

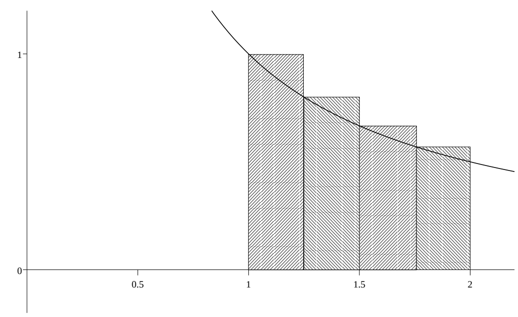
How is $A'(x)$ related to $f(x)$ in problem 1? *same*

How is $B'(x)$ related to $g(x)$ in problem 2? *same*

How is $C'(x)$ related to $h(x)$ in problem 3? *same*

The Natural Logarithm

2a Define $A(x)$ to be the **area** bounded by the x-axis and the function $f(x) = 1/x$ between the line $x = 1$ and the vertical line at x. (See the diagram.)



Based on your work in problem 1,

$A'(x) =$ ___ $1/x$ ___

Compute $A(1) =$ ___ 0 ___

Compute $A(x) =$ ___ $\ln(x)$ ___ *$A(x)$ is an antiderivative of $1/x$ and so must be of the form $\ln(x) + C$ for some constant C. But $0 = A(1) = \ln(1) + C = C$.*

2b So the area under $f(x) = 1/x$ between $x = 1$ and $x = 2$ is equal to $\ln(2)$. Outline this area on the graph. We'll use estimates of this area to compute approximations of $\ln(2)$.

2c Slice the area up into 4 pieces by drawing 3 evenly spaced vertical lines from the x-axis up to the curve.

2d Using the left side of each slice as the height, sketch in 4 rectangles on your graph. What are the x-coordinates of the sides of the rectangles? Plug these x-coordinates into $f(x) = 1/x$ to compute the heights of the rectangles. Find the areas of the 4 rectangles and add them up. This is your first approximation of the area under the curve, and $\ln(2)$. Is it an over-estimate or and under-estimate?

$x_0 = 1,\ x_1 = 5/4,\ x_2 = 3/2,\ x_3 = 7/4,\ x_4 = 2$

$f(1) = 1,\ f(5/4) = 4/5,\ f(3/2) = 2/3,\ f(7/4) = 4/7$

$A_1 = 1 \times 1/4 = 1/4,\ A_2 = 4/5 \times 1/4 = 1/5,\ A_3 = 2/3 \times 1/4 = 1/6,\ A_4 = 4/7 \times 1/4 = 1/7$

$\ln(2) \approx 1/4 + 1/5 + 1/6 + 1/7 \approx 0.7595238$

Overestimate, because rectangles are above the curve.

2e Using the right side of each slice as the height, sketch in 4 rectangles on your graph. Find the area of these rectangles and add them up. This is your second approximation of the area under the curve, and ln(2). Is it an over-estimate or and under-estimate?

$x_0 = 1,\ x_1 = 5/4,\ x_2 = 3/2,\ x_3 = 7/4,\ x_4 = 2$

$f(5/4) = 4/5,\ f(3/2) = 2/3,\ f(7/4) = 4/7,\ f(2) = 1/2$

$A_1 = 4/5 \times 1/4 = 1/5,\ A_2 = 2/3 \times 1/4 = 1/6,\ A_3 = 7/4 \times 1/4 = 1/7,\ A_4 = 1/2 \times 1/4 = 1/8$

$\ln(2) \approx 1/5 + 1/6 + 1/7 + 1/8 \approx 0.6345238$

Underestimate, because rectangles are below the curve.

2f Take the average of your two estimates to get a new estimate of ln(2). How does it compare with the value given by your calculator?

$\ln(2) \approx 0.6970238$

Calculator gives 0.69314718. Good to two decimals.

2g Use the midpoint of each slice to determine the height and sketch in the resulting 4 rectangles. Use them to approximate ln(2). Can you tell if you are getting an over-estimate or and under-estimate? Which of your four estimates gives you the closest answer to the value given by your calculator?

$\bar{x}_1 = 9/8,\ \bar{x}_2 = 11/8,\ \bar{x}_3 = 13/8,\ \bar{x}_4 = 15/8$

$f(9/8) = 8/9,\ f(11/8) = 8/11,\ f(13/8) = 8/13,\ f(15/8) = 8/15$

$A_1 = 8/9 \times 1/4 = 2/9,\ A_2 = 8/11 \times 1/4 = 2/11,\ A_3 = 8/13 \times 1/4 = 2/13,\ A_4 = 8/15 \times 1/4 = 2/15$

$\ln(2) \approx 2/9 + 2/11 + 2/13 + 2/15 \approx 0.69121989$

This one is closest. It's an underestimate, but it's hard to tell without using the calculator.