1. (10 points) Evaluate the following indefinite integrals.

(a) (5 points) We start with substitution \( u = \pi \sqrt{t} \) so that \( du = \frac{\pi}{2\sqrt{t}} \, dx \). The integral becomes

\[
\int \frac{\sin(\pi \sqrt{t})}{\sqrt{t}} \, dt = \frac{2}{\pi} \int \sin(u) \, du = -\frac{2}{\pi} \cos(u) + C = -\frac{2}{\pi} \cos(\pi \sqrt{t}) + C.
\]

(b) (5 points) The substitution \( x = u^6 \) works well since then \( \sqrt{x} = u^3, \sqrt[3]{x} = u^2 \) and \( dx = 6u^5 \, du \). Thus

\[
\int \frac{dx}{\sqrt{x} - \sqrt[3]{x}} = \int \frac{6u^5}{u^3 - u^2} \, du = 6 \int \frac{u^3}{u - 1} \, du.
\]

After doing long division we find that this is equal to

\[
6 \int \left( u^2 + u + \frac{1}{u - 1} \right) \, du = 6 \left[ \frac{u^3}{3} + \frac{u^2}{2} + u + \ln|u - 1| \right] + C.
\]

After substituting back \( u = \sqrt[3]{x} \) we get

\[
2\sqrt[3]{x} + 3\sqrt[3]{x} + 6\sqrt[3]{x} + 6\ln|\sqrt[3]{x} - 1| + C
\]

2. (a) (5 points) (Note: This is Stewart 7.4 #33. This can also be done by partial fractions.) Let \( u = x^4 + 4x^2 + 3 \) so that \( du = (4x^3 + 8x) \, dx = 4(x^3 + 2x) \, dx \). In changing the limits of integration we note that \( x = 0 \implies u = 3 \) and \( x = 1 \implies u = 8 \). Thus

\[
\int_0^1 \frac{x^3 + 2x}{x^4 + 4x^2 + 3} \, dx = \frac{1}{4} \int_3^8 \frac{1}{u} \, du = \frac{1}{4} \ln|u| \bigg|_3^8 = \frac{1}{4} \ln 8 - \frac{1}{4} \ln 3 = \frac{1}{4} \ln \frac{8}{3}
\]

(b) (5 points) (Note: this can also be done by first integrating by parts.)

\[
\int_0^{1/2} \sin^{-1}(x) \, dx
\]

We approach this as a trigonometric substitutition using \( x = \sin \theta \) so that \( dx = \cos \theta \, d\theta \) and \( \theta = \sin^{-1}(x) \). Thus

\[
\int \sin^{-1}(x) \, dx = \int \theta \cos \theta \, d\theta = \theta \sin \theta - \int \sin \theta \, d\theta = \theta \sin \theta + \cos \theta + C = \sin^{-1}(x) \cdot x + \sqrt{1-x^2} + C
\]

where we have used integration by parts to evaluate the \( \theta \) integral. Therefore

\[
\int_0^{1/2} \sin^{-1}(x) \, dx = x \sin^{-1}(x) + \sqrt{1-x^2} \bigg|_0^{1/2} = \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1.
\]
3. (10 points) Let \( R \) be the region that is between the curve \( y = \sqrt{xe^{-x^2}} \) and the \( x \)-axis, is bounded on the left by the line \( x = 1 \), and extends infinitely far out to the right. Let \( S \) be the solid obtained by rotating \( R \) around the \( x \)-axis.

Does \( S \) have finite volume? If so, find it, and give your answer in exact form. (Note: This is Problem #3 the Winter 2014 Final.)

Using the method of disks we see that the volume is given by the improper integral

\[
\int_{1}^{\infty} \pi (\sqrt{xe^{-x^2}})^2 \, dx = \int_{1}^{\infty} \pi xe^{-2x^2} \, dx.
\]

We first evaluate the indefinite integral using a substitution \( u = -2x^2 \) and \( du = -4x \, dx \) so that

\[
\int \pi (\sqrt{xe^{-x^2}})^2 \, dx = \frac{\pi}{4} \int e^u \, du = \frac{\pi}{4} e^u + C = \frac{\pi}{4} e^{-2x^2} + C.
\]

Therefore

\[
\int_{1}^{\infty} \pi (\sqrt{xe^{-x^2}})^2 \, dx = \lim_{t \to \infty} \int_{1}^{t} \pi (\sqrt{xe^{-x^2}})^2 \, dx = \lim_{t \to \infty} \left[ -\frac{\pi}{4} e^{-2t^2} + \frac{\pi}{4} e^{-2} \right] = \frac{\pi}{4} e^{-2}.
\]

4. (10 points) A 1600-lb elevator is suspended by a 200-ft cable that weighs 10 lb/ft. How much work is required to raise the elevator from the basement to the third floor, a distance of 30 ft?

(Note: There are many correct ways to do this problem. As mentioned many times in class weight is a measurement of force so you do not need to multiply by the acceleration due to gravity.)

The work needed to raise the elevator alone is 1600 lb \( \times \) 30 ft = 48,000 ft-lb. The work needed to raise the bottom 170 ft of cable is 170 ft \( \times \) 10 lb/ft \( \times \) 30 ft = 51,000 ft-lb. The work need to raise the top 30 ft of cable is

\[
\int_{30}^{1} \ln x \, dx = \left[ x \ln x - x \right]_{30}^{1} = (3 \ln(3) - 3) - (1 \ln(1) - 1) = 3 \ln(3) - 2
\]

Adding these together, we see that the total work needed is 48,000 + 51,000 + 4,500 = 103,500 ft-lb.

5. (10 points) (Note: This is problem 8 in the Autumn 2009 Final).

(a) (4 points) Use Simpson’s rule with \( n = 4 \) subintervals to approximate the integral. Give your answer in \textit{EXACT} form (involving numbers like \( \ln(3) \), etc.).

\textit{DO NOT GIVE A DECIMAL APPROXIMATION} in this part. \( \Delta x = \frac{3 - 1}{4} = \frac{1}{2} \) so we get

\[
\int_{1}^{3} \ln x \, dx \approx \frac{1}{6} \left[ (0 + 4 \ln(1.5) + 2 \ln(2) + 4 \ln(2.5) + \ln(3)) \right]
\]

(b) (4 points) Compute the integral exactly.

\textit{DO NOT GIVE A DECIMAL APPROXIMATION} in this part.

\[
\int_{1}^{3} \ln x \, dx = \left[ x \ln x - x \right]_{1}^{3} = (3 \ln(3) - 3) - (1 \ln(1) - 1) = 3 \ln(3) - 2
\]

(c) (2 points) Use your calculator to evaluate your answers in part (a) and part (b) as decimals; round your answers to six decimal digits after the decimal point.

\[\text{Simpson} = 1.295322 \quad \text{Exact} = 1.295837\]