Math 125 F - Winter 2016
Midterm Exam Number Two
February 25, 2016

Name: ______________________ Student ID no. : ______________

Signature: ______________________________ Section: ____________

1	21	
2	8 → ~~7~~	
3	12	
4	8 → ~~9~~	
5	5	
6	6	
Total	60	

- This exam consists of SIX problems on SIX pages, including this cover sheet.
- Show all work for full credit.
- You do not need to simplify your answers.
- If you use a trial-and-error or guess-and-check method when a more rigorous method is available, you will not receive full credit.
- If you run out of room, write on the back of the page, but *indicate that you have done so*!
- You may use one hand-written double-sided 8.5″ by 11″ page of notes.
- You may use a *scientific calculator*. Calculators with graphing, differentiation, integration, or algebraic capabilities are not allowed.
- You have 80 minutes to complete the exam.

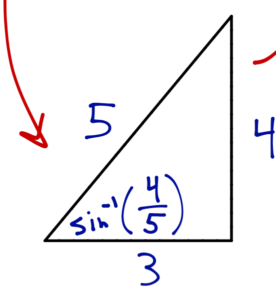
1. **[7 points per part]** Evaluate these integrals!

(a) $$\int_0^4 \frac{dx}{(25-x^2)^{3/2}} = \int_0^{\sin^{-1}\left(\frac{4}{5}\right)} \frac{5\cos\theta\, d\theta}{(25-25\sin^2\theta)^{3/2}} = \int_0^{\sin^{-1}\left(\frac{4}{5}\right)} \frac{5\cos\theta\, d\theta}{(25\cos^2\theta)^{3/2}} = \int_0^{\sin^{-1}\left(\frac{4}{5}\right)} \frac{5\cos\theta\, d\theta}{125\cos^3\theta\, d\theta}$$

$x = 5\sin\theta$ $\qquad$ $\theta = \sin^{-1}\left(\frac{x}{5}\right)$

$dx = 5\cos\theta\, d\theta$

$$= \int_0^{\sin^{-1}\left(\frac{4}{5}\right)} \frac{1}{25}\sec^2\theta\, d\theta = \frac{1}{25}\left(\tan\theta\right)\Big]_0^{\sin^{-1}\left(\frac{4}{5}\right)} = \frac{1}{25}\tan\left(\sin^{-1}\left(\frac{4}{5}\right)\right) = \frac{1}{25}\left(\frac{4}{3}\right) = \boxed{\frac{4}{75}}$$



(b) $$\int_1^\infty \frac{x+\sqrt{x}}{x^3}\, dx$$

$$= \lim_{t\to\infty} \int_1^t \left(x^{-2} + x^{-2.5}\right) dx = \lim_{t\to\infty} \left(-x^{-1} - \frac{x^{-1.5}}{-1.5}\right)\Big]_1^t$$

$$= \lim_{t\to\infty} \left(\underbrace{\frac{-1}{t}}_{0} - \underbrace{\frac{2}{3\sqrt{t^3}}}_{0} + \frac{1}{1} + \frac{2}{3}\right) = \boxed{\frac{5}{3}}$$

Wait, hold up, there's one more part.

(c) $\int x^2 e^{2x}\, dx.$ $= x^2 \frac{e^{2x}}{2} - \int 2xe^{2x}\,dx$ $= \frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \int \frac{1}{2}e^{2x}\,dx$

$u = x^2$ $\quad v = e^{2x}/2$

$du = 2x\,dx$ $\quad dv = e^{2x}\,dx$

$u = x$ $\quad v = e^{2x}/2$

$du = dx$ $\quad dv = e^{2x}\,dx$

$$= \boxed{\frac{x^2 e^{2x}}{2} - \frac{1}{2}xe^{2x} + \frac{e^{2x}}{4} + C}$$

2. **[8 points]** Compute the average value of $f(x) = \tan^6(x)\sec^6(x)$ on the interval $[0, \frac{\pi}{4}]$.

$$\text{Avg. value} = \frac{1}{\frac{\pi}{4} - 0}\int_0^{\pi/4} \tan^6(x)\sec^6(x)\,dx = \frac{4}{\pi}\int_0^{\frac{\pi}{4}} \tan^6(x)\left(\sec^2(x)\right)^2 \sec^2(x)\,dx$$

$$= \frac{4}{\pi}\int_0^{\pi/4} \tan^6(x)\left(1+\tan^2(x)\right)^2 \sec^2(x)\,dx = \frac{4}{\pi}\int_0^1 u^6(1+u^2)^2\,du$$

$u = \tan(x)$

$du = \sec^2(x)\,dx$

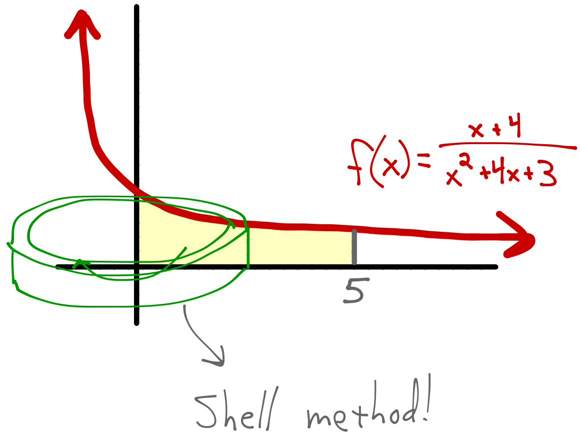
$$= \frac{4}{\pi}\int_0^1 (u^{10} + 2u^8 + u^6)\,du$$

$$= \frac{4}{\pi}\left(\frac{u^{11}}{11} + \frac{2u^9}{9} + \frac{u^7}{7}\right)\Bigg]_0^1$$

$$= \boxed{\frac{4}{\pi}\left(\frac{1}{11} + \frac{2}{9} + \frac{1}{7}\right)}$$

3. **[12 points]** Let $\mathcal{R}$ be the region below the curve $y = \dfrac{x+4}{x^2+4x+3}$ and above the x-axis between $x = 0$ and $x = 5$.

Compute the volume of the solid formed by revolving $\mathcal{R}$ around the y-axis.



$$\text{Volume} = \int_0^5 2\pi x \left(\frac{x+4}{x^2+4x+3}\right) dx$$

$$= \int_0^5 \pi \,\frac{2x^2+8x}{x^2+4x+3}\, dx$$

Long division!

$$\begin{array}{r} 2 \\ x^2+4x+3 \,\big)\, 2x^2+8x \\ -(2x^2+8x+6) \\ \hline -6 \end{array}$$

$$= \pi \int_0^5 \left(2 + \frac{-6}{(x+3)(x+1)}\right) dx$$

$$= \pi \int_0^5 \left(2 + \frac{A}{x+3} + \frac{B}{x+1}\right) dx$$

Partial fractions!

$$\frac{-6}{(x+3)(x+1)} = \frac{A}{x+3} + \frac{B}{x+1}$$

$-6 = A(x+1) + B(x+3)$

$-6 = 2B \leftarrow x = -1$

$-6 = -2A \qquad x = -3$

$A = 3, \quad B = -3$

$$= \pi \int_0^5 \left(2 + \frac{3}{x+3} - \frac{3}{x+1}\right) dx$$

$$= \pi \Big(2x + 3\ln|x+3| - 3\ln|x+1| \Big) \Big]_0^5$$

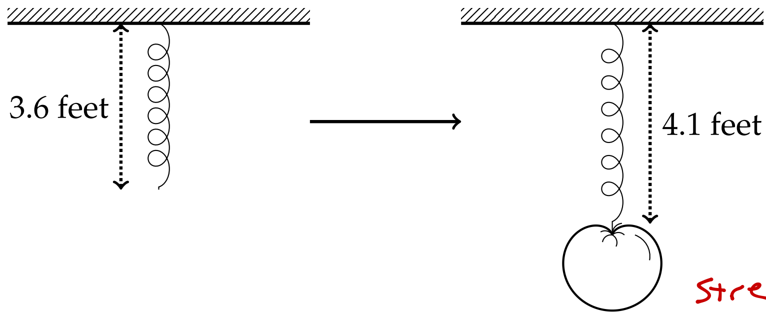
$$= \pi \Big(10 + 3\ln(8) - 3\ln(6) - 3\ln(3) + \underbrace{3\ln(1)}_{0} \Big)$$

$$= \boxed{\pi \left(10 + 3\ln\left(\frac{4}{9}\right)\right)}$$

4. **[8 points]** Recall Hooke's law, which says that the force required to compress or stretch a spring from its natural length by some distance is proportional to that distance.

A spring (of negligible mass) is suspended from the ceiling and has a natural length of 3.6 feet. When a 0.4-pound tomato is attached to the end of the spring, it stretches to a length of 4.1 feet.

Compute the work required to stretch this same spring from a length of 5 feet to 6 feet. (Express your answer in foot-pounds.)



Stretched 1.4 ft

Stretched 2.4 ft

$$F = kx$$

$$0.4\ \text{lb} = k(0.5\ \text{ft})$$

$$k = 0.8\ \text{lb}/\text{ft}$$

$$W = \int_{1.4}^{2.4} \underbrace{(0.8)x}_{\text{force}}\ \underbrace{dx}_{\text{dist}} = \left(0.4x^2\right)\Big]_{1.4}^{2.4} = 0.4\left(2.4^2 - 1.4^2\right)$$

$$= \boxed{1.52\ \text{foot-pounds}}$$

5. **[5 points]** Use Simpson's rule with $n = 6$ subintervals to approximate $\int_2^5 \sin(x^2)\,dx$.

Please leave your answer in exact form.

$x_0 = 2$ $\quad x_1 = 2.5$ $\quad x_2 = 3$ $\quad x_3 = 3.5$ $\quad x_4 = 4$ $\quad x_5 = 4.5$ $\quad x_6 = 5$ $\quad \Delta x = \frac{1}{2}$

$$\boxed{\frac{1}{6}\left(\sin(2^2) + 4\sin(2.5^2) + 2\sin(3^2) + 4\sin(3.5^2) + 2\sin(4^2) + 4\sin(4.5^2) + \sin(5^2)\right)}$$

6. **[6 points]** Determine whether $\int_0^5 \frac{e^x + \sin^2(x)}{x^2}\,dx$ converges or diverges.

(You do *not* need to evaluate the integral.)

$e^x \geq 1$ and $\sin^2(x) \geq 0$ on $[0, 5]$, so $\frac{e^x + \sin^2(x)}{x^2} \geq \frac{1}{x^2}$

Now, $\int_0^5 \frac{1}{x^2} = \lim_{t \to 0^+} \int_t^5 \frac{1}{x^2}\,dx$

$= \lim_{t \to 0^+} \left(\frac{-1}{x}\right)\Big]_t^5 = \lim_{t \to 0^+} \left(\frac{-1}{5} + \frac{1}{t}\right) = \infty$

So $\int_0^5 \frac{1}{x^2}\,dx$ diverges,

which means the bigger integral $\boxed{\text{also diverges.}}$