Math 125 F - Winter 2016
Midterm Exam Number Two
February 25, 2016

Name: ____________________________  Student ID no.: __________________

Signature: _________________________  Section: ____________

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- This exam consists of SIX problems on SIX pages, including this cover sheet.
- Show all work for full credit.
- You do not need to simplify your answers.
- If you use a trial-and-error or guess-and-check method when a more rigorous method is available, you will not receive full credit.
- If you run out of room, write on the back of the page, but indicate that you have done so!
- You may use one hand-written double-sided 8.5” by 11” page of notes.
- You may use a scientific calculator. Calculators with graphing, differentiation, integration, or algebraic capabilities are not allowed.
- You have 80 minutes to complete the exam.
1. **[7 points per part]** Evaluate these integrals!

(a) \( \int_0^4 \frac{dx}{(25 - x^2)^{3/2}} = \int_0^{\sin^{-1} \left( \frac{4}{5} \right)} \frac{5 \cos \theta d\theta}{(25 - 25 \sin^2 \theta)^{3/2}} = \int_0^{\sin^{-1} \left( \frac{4}{5} \right)} \frac{5 \cos \theta d\theta}{125 \cos^2 \theta} \)

\[ x = 5 \sin \theta \]
\[ \theta = \sin^{-1} \left( \frac{x}{5} \right) \]

\[ \sin^{-1} \left( \frac{4}{5} \right) \]

\[ d\theta = 5 \cos \theta \, d\theta \]

\[ \int_0^{\sin^{-1} \left( \frac{4}{5} \right)} \frac{5 \cos \theta d\theta}{125} \left( \tan \Theta \right) \left[ \right]_0 \]

\[ = \frac{1}{25} \left( \tan \left( \sin^{-1} \left( \frac{4}{5} \right) \right) \right) = \frac{1}{25} \left( \frac{4}{3} \right) = \frac{4}{75} \]

(b) \( \int_1^{\infty} \frac{x + \sqrt{x}}{x^3} \, dx \)

\[ = \lim_{t \to \infty} \int_1^t \left( -2 + x^{1/2} \right) \, dx = \lim_{t \to \infty} \left[ -x^{-1} + \frac{x^{3/2}}{3/2} \right]_1^t \]

\[ = \lim_{t \to \infty} \left( \frac{-1}{t} - \frac{2}{3 \sqrt{t}} + \frac{1}{1} + \frac{2}{3} \right) = \left[ \frac{5}{3} \right] \]
Wait, hold up, there’s one more part.

(c) \[ \int x^2 e^{2x} \, dx \quad = \quad \frac{2x}{2} - \frac{1}{2} e^{2x} + \frac{e^2}{4} \]

\[ u = x \quad \quad v = e^{2x} \]

\[ du = dx \quad \quad dv = e^{2x} \, dx \]

\[ \int x^2 e^{2x} \, dx = \frac{2x}{2} e^{2x} - \frac{1}{2} e^{2x} + \frac{e^2}{4} + C \]

2. [8 points] Compute the average value of \( f(x) = \tan^6(x) \sec^6(x) \) on the interval \([0, \frac{\pi}{4}]\).

\[ \text{Avg. value} = \frac{1}{\frac{\pi}{4}} \int_{0}^{\frac{\pi}{4}} \tan^6(x) \sec^6(x) \, dx = \frac{4}{\pi} \int_{0}^{\frac{\pi}{4}} \tan^6(x) \left( \sec^2(x) \right)^2 \sec^2(x) \, dx \]

\[ = \frac{4}{\pi} \int_{0}^{\frac{\pi}{4}} \tan^6(x) \left( 1 + \tan^2(x) \right)^2 \sec^2(x) \, dx = \frac{4}{\pi} \int_{0}^{1} u^6 (1 + u^2)^2 \, du \]

\[ u = \tan(x) \]
\[ dv = \sec^2(x) \, dx \]

\[ = \frac{4}{\pi} \left[ \left. \frac{u^{11}}{11} + 2u^7 + u^9 \right|_0^1 \right] \]

\[ = \frac{4}{\pi} \left( \frac{1}{11} + \frac{2}{9} + \frac{1}{7} \right) \]

\[ = \frac{4}{\pi} \left( \frac{1}{11} + \frac{2}{9} + \frac{1}{7} \right) \]
3. [12 points] Let \( R \) be the region below the curve \( y = \frac{x + 4}{x^2 + 4x + 3} \) and above the \( x \)-axis between \( x = 0 \) and \( x = 5 \).

Compute the volume of the solid formed by revolving \( R \) around the \( y \)-axis.

\[
\text{Volume} = \int_0^5 2\pi x \left( \frac{x + 4}{x^2 + 4x + 3} \right) \, dx
\]

\[
= \int_0^5 2\pi \frac{2x^2 + 8x}{x^2 + 4x + 3} \, dx
\]

\[
= \pi \int_0^5 \left( 2 + \frac{-6}{(x+3)(x+1)} \right) \, dx
\]

\[
= \pi \int_0^5 \left( -2 + \frac{A}{x+3} + \frac{B}{x+1} \right) \, dx
\]

\[
= \pi \left[ (-2 + \frac{3}{x+3} - \frac{3}{x+1}) \right]_0^5
\]

\[
= \pi \left( -2 + 3 \ln(5) - 3 \ln(6) - 3 \ln(3) + 3 \ln(1) \right)
\]

\[
= \pi \left( -10 + 3 \ln\left( \frac{5}{6} \right) \right)
\]
4. [8 points] Recall Hooke’s law, which says that the force required to compress or stretch a spring from its natural length by some distance is proportional to that distance.

A spring (of negligible mass) is suspended from the ceiling and has a natural length of 3.6 feet. When a 0.4-pound tomato is attached to the end of the spring, it stretches to a length of 4.1 feet.

Compute the work required to stretch this same spring from a length of 5 feet to 6 feet. (Express your answer in foot-pounds.)

\[
F = kx
\]

\[0.4 \, \text{lb} = k \cdot (0.5 \, \text{ft})\]

\[k = \frac{0.4 \, \text{lb}}{0.5 \, \text{ft}} = 0.8 \, \text{lb/ft}\]

\[W = \int_{1.4}^{2.4} (0.8) \, dx = (0.4x^2) \bigg|_{1.4}^{2.4} = 0.4(2.4^2 - 1.4^2) = 1.52 \, \text{foot-pounds}\]
5. **[5 points]** Use Simpson’s rule with \( n = 6 \) subintervals to approximate \( \int_{2}^{5} \sin(x^2) \, dx \).

Please leave your answer in exact form.

\[
\Delta x = \frac{1}{2}
\]

\[
\frac{1}{6} \left( \sin(2^2) + 4\sin(2.5^2) + 2\sin(3^2) + 4\sin(3.5^2) + 2\sin(4^2) + 4\sin(4.5^2) + \sin(5^2) \right)
\]

6. **[6 points]** Determine whether \( \int_{0}^{5} \frac{e^x + \sin^2(x)}{x^2} \, dx \) converges or diverges.

(You do not need to evaluate the integral.)

\[e^x \geq 1 \text{ and } \sin^2(x) \geq 0 \quad \text{on } [0, 5]\]

So

\[\frac{e^x + \sin^2(x)}{x^2} \geq \frac{1}{x^2}\]

Now,

\[\int_{0}^{5} \frac{1}{x^2} \, dx \sim \int_{t}^{5} \frac{1}{x^2} \, dx\]

\[= \lim_{t \to 0^+} \left( \frac{-1}{x} \right) \bigg|_{t}^{5} = \lim_{t \to 0^+} \left( \frac{-1}{5} + \frac{1}{t} \right) = \infty\]

So \( \int_{0}^{5} \frac{1}{x^2} \, dx \) diverges,

which means the bigger integral also diverges.