1 (12 points) Compute the following indefinite integrals.

(a) (6 points) $\int \sin^5\theta\cos^3\theta\,d\theta$

$$\begin{aligned}
\int \sin^5\theta\cos^3\theta\,d\theta &= \int \sin^5\theta\cos^2\theta\ \cos\theta\,d\theta \\
&= \int \sin^5\theta(1-\sin^2\theta)\ \cos\theta\,d\theta \qquad \text{let } u=\sin\theta \\
&= \int u^5(1-u^2)\,du \\
&= \frac{1}{6}u^6 - \frac{1}{8}u^8 + C \\
&= \frac{1}{6}\sin^6\theta - \frac{1}{8}\sin^8\theta + C
\end{aligned}$$

(b) (6 points) $\int \frac{1}{y\sqrt{y^2-25}}\,dy$

Let $y = 5\sec\theta$. *Then* $dy = 5\sec\theta\tan\theta\,d\theta$ *and* $\sqrt{y^2-25} = 5\tan\theta$

$$\begin{aligned}
\int \frac{1}{y\sqrt{y^2-25}}\,dy &= \int \frac{1}{5\sec\theta\cdot 5\tan\theta}\,5\sec\theta\tan\theta\,d\theta \\
&= \int \frac{1}{5}\,d\theta \\
&= \frac{1}{5}\theta + C \\
&= \frac{1}{5}\tan^{-1}\frac{1}{5}\sqrt{y^2-25} + C
\end{aligned}$$

2 (12 points) Compute the following definite integrals. Give your answers in exact form.

(a) (6 points) $\displaystyle\int_3^5 \frac{5x^2}{x^2-3x+2}\,dx$

First use Partial Fractions to write

$$\frac{5x^2}{x^2-3x+2} = 5 + \frac{20}{x-2} - \frac{5}{x-1}$$

$$\begin{aligned}
\int_3^5 \frac{5x^2}{x^2-3x+2}\,dx &= \int_3^5 5 + \frac{20}{x-2} - \frac{5}{x-1}\,dx \\
&= 5x + 20\ln|x-2| - 5\ln|x-1|\Big|_3^5 \\
&= 10 + 20\ln 3 - 5\ln 4 + 5\ln 2 \\
&= 10 + 20\ln 3 - 5\ln 2
\end{aligned}$$

(b) (6 points) $\displaystyle\int_0^1 t\,\sin^{-1} t\,dt$

First use Integration by Parts with $u = \sin^{-1} t$ *and* $dv = t\,dt$ *to get*

$$\begin{aligned}
\int_0^1 t\,\sin^{-1} t\,dt &= \frac{1}{2}t^2\sin^{-1} t\Big|_0^1 - \frac{1}{2}\int_0^1 \frac{t^2}{\sqrt{1-t^2}}\,dt \\
&= \frac{\pi}{4} - \frac{1}{2}\int_0^1 \frac{t^2}{\sqrt{1-t^2}}\,dt
\end{aligned}$$

To solve $\displaystyle\int \frac{t^2}{\sqrt{1-t^2}}\,dt$, *set* $t = \sin\theta$ *so that* $dt = \cos\theta\,d\theta$ *and* $\sqrt{1-t^2} = \cos\theta$.

$$\begin{aligned}
\int \frac{t^2}{\sqrt{1-t^2}}\,dt &= \int \frac{\sin^2\theta}{\cos\theta}\cos\theta\,d\theta \\
&= \int \sin^2\theta\,d\theta \\
&= \frac{1}{2}\int 1 - \cos 2\theta\,d\theta \\
&= \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + C \\
&= \frac{1}{2}\theta - \frac{1}{2}\sin\theta\cos\theta + C \\
&= \frac{1}{2}\sin^{-1} t - \frac{1}{2}t\sqrt{1-t^2} + C
\end{aligned}$$

Thus

$$\int_0^1 \frac{t^2}{\sqrt{1-t^2}}\,dt = \frac{1}{2}\sin^{-1} t - \frac{1}{2}t\sqrt{1-t^2}\Big|_0^1 = \frac{\pi}{4}$$

and

$$\int_0^1 t\,\sin^{-1} t\,dt = \frac{\pi}{4} - \frac{1}{2}\cdot\frac{\pi}{4} = \frac{\pi}{8}$$

3 (8 points) A rope is used to pull a bucket full of water up from a well that is 10 m deep. The rope has a total mass of 5 kg. The bucket of water has a mass of 11 kg. The acceleration due to gravity is 9.8 m/sec^2. Set up an integral that computes the work done in lifting the bucket all the way up. **Do not simplify or evaluate the integral.**

Choose a coordinate system with $y = 0$ at the top of the well and $y = 10$ at the bottom.

Let $W(y)$ be the weight of the load at depth y meters.

Then the integral we want is $\int_0^{10} W(y)\,dy$.

The weight of the bucket is $9.8 \times 11 = 107.8$ *Newtons.*

The rope has a mass of 0.5 kg/m. When the load is at depth y meters, we are lifting y meters of rope. This weighs $9.8 \times 0.5y = 4.9y$ *Newtons.*

Thus the total weight of the load is $W(y) = 107.8 + 4.9y$ *Newtons*

The total work is given by $\int_0^{10} 107.8 + 4.9y\,dy$.

4 (8 points) Use the Trapezoid Rule with $n = 5$ to approximate the average value of the function $\phi(x) = \sin(1/x)$ on the interval $x = 1$ to $x = 4$. Round your answer to 3 decimal places.

We must approximate $\frac{1}{4-1}\int_1^4 \phi(x)\,dx$ *where* $\phi(x) = \sin(1/x)$.

$\Delta x = \frac{4-1}{5} = 0.6$

$x_0 = 1,\ x_1 = 1.6,\ x_2 = 2.2,\ x_3 = 2.8,\ x_4 = 3.4,\ x_5 = 4$

$$\begin{aligned}
\int_1^4 \phi(x)\,dx &\approx \frac{1}{2}\cdot \Delta x \cdot [\phi(x_1) + 2\phi(x_2) + 2\phi(x_3) + 2\phi(x_4) + \phi(x_5)] \\
&= 0.3 \cdot [\phi(1) + 2\phi(1.6) + 2\phi(2.2) + 2\phi(2.8) + 2\phi(3.4) + \phi(4)] \\
&\approx 0.3 \times 4.416166 \\
&= 1.32485
\end{aligned}$$

The average value is $\frac{1}{3}\int_1^4 \phi(x)\,dx \approx \frac{1}{3} \times 1.32485 \approx 0.442$

5 (10 points) Determine if the improper integral $\displaystyle\int_{-1}^{0} \frac{e^{1/t}}{t^3}\,dt$ is convergent or divergent. If it is convergent, evaluate it.

Step 1: Calculate $\displaystyle\int \frac{e^{1/t}}{t^3}\,dt.$

First make the substitution $x = \dfrac{1}{t}$. *Then* $dx = -\dfrac{1}{t^2}\,dt.$

The integral transforms to $\displaystyle\int -xe^x\,dx.$

Now use Integration by Parts with $u = -x$ *and* $dv = e^x\,dx$ *to get*

$$\int -xe^x\,dx = -xe^x + e^x + C = (1-x)e^x + C$$

Thus $\displaystyle\int \frac{e^{1/t}}{t^3}\,dt = \left(1 - \frac{1}{t}\right)e^{1/t} + C.$

Step 2: Compute $\displaystyle\lim_{b\to 0^-}\int_{-1}^{b} \frac{e^{1/t}}{t^3}\,dt.$

$$\begin{aligned}
\lim_{b\to 0^-}\int_{-1}^{b} \frac{e^{1/t}}{t^3}\,dt &= \lim_{b\to 0^-}\left(1-\frac{1}{t}\right)e^{1/t}\Big|_{-1}^{b} && \textit{by Step 1}\\
&= \lim_{b\to 0^-}\left(1-\frac{1}{b}\right)e^{1/b} - \frac{2}{e}\\
&= \lim_{b\to 0^-}\frac{1-\frac{1}{b}}{e^{-1/b}} - \frac{2}{e} && \tfrac{\infty}{\infty}\ \textit{l'Hôpital's Rule}\\
&= \lim_{b\to 0^-}\frac{1/b^2}{1/b^2\cdot e^{-1/b}} - \frac{2}{e}\\
&= \lim_{b\to 0^-} e^{1/b} - \frac{2}{e}\\
&= -\frac{2}{e}
\end{aligned}$$