Math 125 E and F **Midterm 2 Solutions** **Burdzy, Autumn 2015**

Your Name

Your Signature

Student ID #

Circle quiz section and print TA's name:

EA EB EC ED FA FB FC FD

- Turn off all cell phones, pagers, radios, mp3 players, and other similar devices.
- This exam is closed book. You may use one $8\frac{1}{2}$" $\times$ 11" sheet of handwritten notes (both sides).
- You can use only Texas Instruments TI-30X calculator.
- Give your answers in exact form, not decimals, unless stated otherwise.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct.
- **Check your work carefully.** We will award only limited partial credit.
- Place a box around your answer to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Raise your hand if you have a question.
- This exam has 6 pages, plus this cover sheet. Make sure that your exam is complete.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
Total	60	

1. (10 points) Compute $\int x\sin(4x)\cos(3x)\,dx$.

Solution:

Recall that

$$\sin(4x)\cos(3x) = \frac{1}{2}(\sin(4x+3x)+\sin(4x-3x)) = \frac{1}{2}(\sin(7x)+\sin x).$$

Hence

$$\int x\sin(4x)\cos(3x)\,dx = \frac{1}{2}\int x(\sin(7x)+\sin x)\,dx = \frac{1}{2}\int x\sin(7x)\,dx + \frac{1}{2}\int x\sin x\,dx.$$

We compute the second integral using integration by parts.

$$\frac{1}{2}\int x\sin(x)\,dx = -\frac{1}{2}x\cos x + \frac{1}{2}\int \cos x dx = -\frac{1}{2}x\cos x + \frac{1}{2}\sin x + C$$

The first integral can be transformed into the second one by the substitution $u = 7x$ with $du/7 = dx$,

$$\frac{1}{2}\int x\sin(7x)\,dx = \frac{1}{2}\int \frac{u}{7}\sin(u)\,du/7 = \frac{1}{49}\cdot\frac{1}{2}\int u\sin(u)\,du = \frac{1}{49}\left(-\frac{1}{2}u\cos u + \frac{1}{2}\sin u\right) + C_1$$

$$= \frac{1}{49}\left(-\frac{7}{2}x\cos(7x) + \frac{1}{2}\sin(7x)\right) + C_1$$

We add the two partial results to obtain

$$\int x\sin(4x)\cos(3x)\,dx = \boxed{-\frac{1}{14}x\cos(7x) + \frac{1}{98}\sin(7x) - \frac{1}{2}x\cos x + \frac{1}{2}\sin x + C_2.}$$

2. (10 points) Determine whether the improper integral $\int_0^\infty e^{-x}\sin x\,dx$ is convergent. If it is convergent, find its value.

Solution:

We will find the indefinite integral $\int e^{-x}\sin x\,dx$ using integration by parts.

$$\int e^{-x}\sin x\,dx = -e^{-x}\cos x - \int e^{-x}\cos x dx$$

We integrate by parts again.

$$\int e^{-x}\sin x\,dx = -e^{-x}\cos x - \int e^{-x}\cos x dx = -e^{-x}\cos x - e^{-x}\sin x - \int e^{-x}\sin x dx$$

We take the term $-\int e^{-x}\sin x dx$ to the left hand side.

$$2\int e^{-x}\sin x\,dx = -e^{-x}\cos x - e^{-x}\sin x + C$$

$$\int e^{-x}\sin x\,dx = -\frac{1}{2}e^{-x}(\cos x + \sin x) + C_1$$

We have

$$\lim_{b\to\infty}\int_0^b e^{-x}\sin x\,dx = \lim_{b\to\infty}\left(-\frac{1}{2}e^{-x}(\cos x + \sin x)\Big|_0^b\right) = \frac{1}{2}e^{-0}(\cos 0 + \sin 0) = \boxed{\frac{1}{2}}$$

REMARK: The comparison test cannot be used because the function $e^{-x}\sin x$ changes sign infinitely many times as x goes to infinity.

3. (10 points) Compute $\int \frac{x^5}{\sqrt{x^2+1}}\,dx$.

Solution:

Method I. We make the substitution $u = x^2+1$, $du = 2xdx$. Then $x^2 = u-1$ and

$$\int \frac{x^5}{\sqrt{x^2+1}}\,dx = \int \frac{(u-1)^2}{u^{1/2}}\frac{1}{2}du = \frac{1}{2}\int \frac{u^2-2u+1}{u^{1/2}}du = \frac{1}{2}\int (u^{3/2}-2u^{1/2}+u^{-1/2})du$$

$$= \frac{1}{2}\left(\frac{2}{5}u^{5/2} - \frac{4}{3}u^{3/2} + 2u^{1/2}\right) + C = \frac{1}{2}u^{1/2}\left(\frac{2}{5}u^2 - \frac{4}{3}u + 2\right) + C$$

$$= \frac{1}{2}\sqrt{x^2+1}\left(\frac{2}{5}(x^4+2x^2+1) - \frac{4}{3}(x^2+1) + 2\right) + C = \boxed{\frac{1}{15}\sqrt{x^2+1}(3x^4-4x^2+8)+C}$$

Method II. We apply inverse trigonometric substitution $x = \tan\theta$. Then $dx = \sec^2\theta d\theta$ and

$$\int \frac{x^5}{\sqrt{x^2+1}}\,dx = \int \frac{\tan^5\theta}{\sqrt{\tan^2\theta+1}}\sec^2\theta d\theta = \int \frac{\tan^5\theta}{\sec\theta}\sec^2\theta d\theta = \int \tan^5\theta \sec\theta d\theta$$

$$= \int \frac{\sin^5\theta}{\cos^5\theta}\cdot\frac{1}{\cos\theta}d\theta = \int \frac{\sin^4\theta}{\cos^6\theta}\sin\theta d\theta = \int \frac{(1-\cos^2\theta)^2}{\cos^6\theta}\sin\theta d\theta$$

We apply another substitution $u = \cos\theta$, $du = -\sin\theta d\theta$.

$$\int \frac{(1-\cos^2\theta)^2}{\cos^6\theta}\sin\theta d\theta = -\int \frac{(1-u^2)^2}{u^6}du = -\int \frac{1-2u^2+u^4}{u^6}du = -\int (u^{-6}-2u^{-4}+u^{-2})du$$

$$= -\left(\frac{1}{-5}u^{-5} - 2\cdot\frac{1}{-3}u^{-3} + \frac{1}{-1}u^{-1}\right) + C = u^{-5}/5 - 2u^{-3}/3 + u^{-1} + C$$

Since $x = \tan\theta$ so

$$u^{-1} = 1/u = 1/\cos\theta = \sec\theta = \sqrt{\tan^2\theta+1} = \sqrt{x^2+1} = (x^2+1)^{1/2}.$$

Hence,

$$= u^{-5}/5 - 2u^{-3}/3 + u^{-1} + C = \frac{1}{5}(x^2+1)^{5/2} - \frac{2}{3}(x^2+1)^{3/2} + (x^2+1)^{1/2} + C$$

$$= \boxed{\frac{1}{15}\sqrt{x^2+1}(3x^4-4x^2+8)+C}$$

4. (10 points) Find the length of the curve $y = \ln(1-x^2)$ between $x = 0$ and $x = 1/8$. Give answer in exact simplified form.

Solution: The length of the curve $y = f(x) = \ln(1-x^2)$ between $x = 0$ and $x = 1/8$ is equal to

$$\int_0^{1/8} \sqrt{1+(f'(x))^2}\,dx.$$

We have $f'(x) = -2x/(1-x^2)$ so

$$1+(f'(x))^2 = 1+\frac{4x^2}{(1-x^2)^2} = \frac{(1+x^2)^2}{(1-x^2)^2}$$

and, therefore,

$$\int_0^{1/8} \sqrt{1+(f'(x))^2}\,dx = \int_0^{1/8} \sqrt{\frac{(1+x^2)^2}{(1-x^2)^2}}\,dx = \int_0^{1/8} \frac{1+x^2}{1-x^2}\,dx$$

$$= \int_0^{1/8} \left(-1+\frac{1}{1+x}+\frac{1}{1-x}\right) dx = (-x+\ln(1+x)-\ln(1-x))\Big|_0^{1/8}$$

$$= -1/8+\ln(9/8)-\ln(7/8) = \boxed{-1/8+\ln(9/7)}$$

5. (10 points) Use Simpson's rule with $n = 4$ subintervals to find an approximation to $\int_{\pi/2}^{3\pi/2} \frac{\sin x}{x} dx$. Give the answer in exact form or as a decimal number with at least 3 significant digits.

Solution: With $f(x) = \frac{\sin x}{x}$, we have

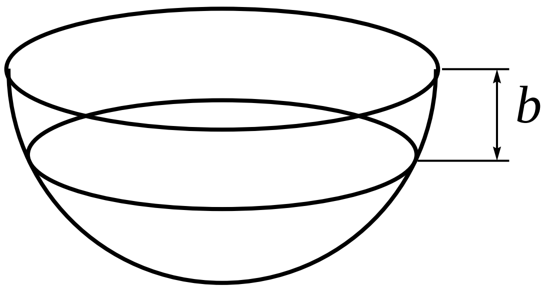
$$\int_{\pi/2}^{3\pi/2} \frac{\sin x}{x} dx \approx \frac{\pi}{4} \cdot \frac{1}{3} (f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4))$$

$$= \frac{\pi}{4} \cdot \frac{1}{3} \left(\frac{\sin x_0}{x_0} + 4\frac{\sin x_1}{x_1} + 2\frac{\sin x_2}{x_2} + 4\frac{\sin x_3}{x_3} + \frac{\sin x_4}{x_4} \right)$$

$$= \frac{\pi}{4} \cdot \frac{1}{3} \left(\frac{\sin(\pi/2)}{\pi/2} + 4\frac{\sin(3\pi/4)}{3\pi/4} + 2\frac{\sin \pi}{\pi} + 4\frac{\sin(5\pi/4)}{5\pi/4} + \frac{\sin(3\pi/2)}{3\pi/2} \right) = \boxed{\frac{1}{45} \left(4\sqrt{2} + 5\right)} \approx 0.236819$$

6. A container has the shape of a hemisphere with radius 2 meters. The container was completely filled with water. The water was pumped out over the edge in two stages. At the first stage, only the top layer of water of depth b meters was pumped out. At the second stage, the remaining water was pumped out. The amount of work done at the first stage was the same as the amount of work done at the second stage.
   (a) (5 points) Express the amount of work done at the first stage as a function of b. Simplify your answer.
   (b) (5 points) Find b. Give the answer as an exact number or in decimal form with at least 3 significant digits.

   The mass density of water is 1,000 kg per cubic meter. Assume that the gravitational acceleration is 9.8 meters per second squared.



Solution:

(a) Let the top of the container play the role of the 0 level. The volume of a layer of water y meters below the top is $\pi r^2 \Delta y = \pi(4-y^2)\Delta y$. The work needed to pump out this layer of water is $1,000 \cdot 9.8\pi(4-y^2)\Delta y \cdot y$. The amount of work needed to pump out the top layer of water of depth b is equal to

$$\int_0^b 1,000 \cdot 9.8\pi(4-y^2)y\,dy = 9,800\pi(2y^2 - y^4/4)\Big|_0^b = \boxed{9,800\pi(2b^2 - b^4/4)} \text{ (joules)}$$

(b) The total work done at both stages of pumping was

$$\int_0^2 1,000 \cdot 9.8\pi(4-y^2)y\,dy = 9,800\pi(2y^2 - y^4/4)\Big|_0^2 = 9,800\pi(2 \cdot 2^2 - 2^4/4) = 4 \cdot 9,800\pi.$$

Since the work done at the first stage was $1/2$ of this amount, we have

$$9,800\pi(2b^2 - b^4/4) = 4 \cdot 9,800\pi/2.$$

$$2b^2 - b^4/4 = 2$$

Let $a = b^2$.

$$2a - a^2/4 = 2$$

$$a^2 - 8a + 8 = 0$$

$$a = \frac{8 \pm \sqrt{8^2 - 4 \cdot 8}}{2} = 4 - 2\sqrt{2}$$

We chose the value of a less than 4 because $b = \sqrt{a}$ must be less than 2.

$$b = \sqrt{a} = \boxed{\sqrt{4 - 2\sqrt{2}}} \approx 1.08239 \text{ (meters)}$$