Math 125, Sections C and F, Fall 2014, Midterm II

November 13, 2014

Name Solutions

TA/Section

Instructions.

- There are 4 questions. The exam is out of 40 points.
- You are allowed to use one page of notes written only on one side of the sheet in your own handwriting. Hand in you note sheet with your exam.
- Calculators are NOT allowed. Put away ALL electronic devices.
- For your integrals you may use the following formulas. Anything else must be justified by your work.

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, n \neq -1 \qquad \int e^x\,dx = e^x + C \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C \qquad \int \sec^2 x\,dx = \tan x + C$$

$$\int \csc^2 x\,dx = -\cot x + C \qquad \int \csc x \cot x\,dx = -\csc x + C \qquad \int \sec x \tan x\,dx = \sec x + C$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{a^2 + x^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C \qquad \int \sec x\,dx = \ln|\sec x + \tan x| + C$$

- **Show your work**. If I cannot read or follow your work, I cannot grade it. You may not get full credit for a right answer if your answer is not justified by your work. If you continue at the back of a page, make a note for me.

Question	points
1	
2	
3	
4	
Total	

1. (12 points) Evaluate the following indefinite integrals.

(a) $\int \frac{3x+5}{\sqrt{x+2}}dx$

$u = x+2 \longrightarrow x = u-2$

$du = dx$

$= \int \frac{3(u-2)+5}{\sqrt{u}} du$

$= \int 3u^{1/2} - u^{-1/2} du$

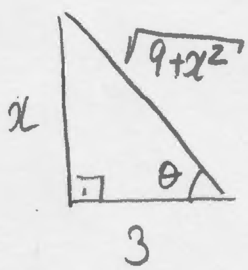
$= 2u^{3/2} - 2u^{1/2} + C$

$= 2(x+2)^{3/2} - 2\sqrt{x+2} + C$

(b) $\int \frac{x^2}{(x^2+9)^{5/2}}dx$

$x = 3\tan\theta$

$dx = 3\sec^2\theta \, d\theta$



$\int \frac{9\tan^2\theta}{(3\sec\theta)^5} 3\sec^2\theta \, d\theta$

$= \frac{1}{9}\int \frac{\tan^2\theta}{\sec^3\theta} d\theta = \frac{1}{9}\int \sin^2\theta \cos\theta \, d\theta$

$= \frac{1}{27} \sin^3\theta + C$

$= \frac{1}{27}\left(\frac{x}{\sqrt{9+x^2}}\right)^3 + C$

2. (12 points) Evaluate the following integrals.

(a) $\int_1^3 \frac{\arctan(\sqrt{x})}{\sqrt{x}} dx$ $\qquad w = \sqrt{x} \qquad dw = \frac{1}{2\sqrt{x}} dx$

$$= 2\int_1^{\sqrt{3}} \arctan(w)\,dw \qquad u = \arctan(w) \quad dv = dw$$

$$du = \frac{1}{1+w^2} dw \qquad v = w$$

$$= 2\left[w\arctan(w)\Big|_1^{\sqrt{3}} - \int_1^{\sqrt{3}} \frac{w}{1+w^2} dw \right]$$

$$= 2\left[\sqrt{3}\cdot\frac{\pi}{3} - \frac{\pi}{4} - \frac{1}{2}\ln(1+w^2)\Big|_1^{\sqrt{3}} \right] = 2\left[\frac{\pi\sqrt{3}}{3} - \frac{\pi}{4} - \frac{1}{2}\ln 2 \right]$$

(b) $\int_0^\infty \frac{e^{3x}}{4+e^{6x}} dx$

$$= \lim_{t\to\infty} \int_0^t \frac{e^{3x}}{4+(e^{3x})^2} dx \qquad u = e^{3x} \quad du = 3e^{3x} dx$$

$$= \lim_{t\to\infty} \frac{1}{3} \int_1^{e^{3t}} \frac{du}{4+u^2}$$

$$= \lim_{t\to\infty} \frac{1}{3} \left[\frac{1}{2}\arctan\left(\frac{u}{2}\right)\Big|_1^{e^{3t}} \right]$$

$$= \frac{1}{6} \lim_{t\to\infty} \left[\arctan\left(\frac{e^{3t}}{2}\right) - \arctan\left(\frac{1}{2}\right) \right]$$

$$= \frac{1}{6}\left[\frac{\pi}{2} - \arctan\left(\frac{1}{2}\right) \right]$$

3. (6 points) Estimate the length of the curve

$$y = x + \frac{1}{3}x^3, \qquad -3 \le x \le 3$$

using Simpson's Rule with $n = 6$. Simplify your answer.

$$y' = 1 + x^2$$

$$L = \int_{-3}^{3} \sqrt{1 + (1+x^2)^2}\, dx$$

$$= \int_{-3}^{3} \sqrt{2 + 2x^2 + x^4}\, dx$$

$$\Delta x = \frac{3-(-3)}{6} = 1$$

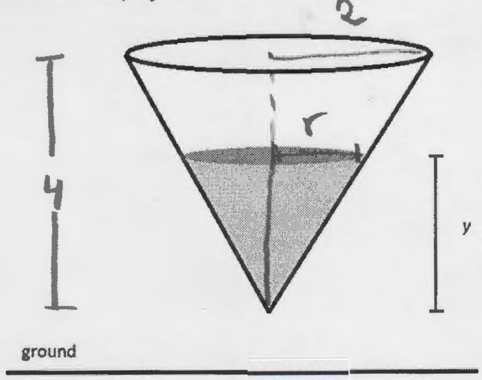
$$L \simeq S_6 = \frac{1}{3}\left[f(-3) + 4f(-2) + 2f(-1) + 4f(0) + 2f(1) + 4f(2) + f(3) \right]$$

$$= \frac{1}{3}\left[\sqrt{101} + 4\sqrt{26} + 2\sqrt{5} + 4\sqrt{2} + 2\sqrt{5} + 4\sqrt{26} + \sqrt{101} \right]$$

$$= \frac{1}{3}\left[2\sqrt{101} + 8\sqrt{26} + 4\sqrt{5} + 4\sqrt{2} \right]$$

4. (10 points) A tank is in the shape of a cone with its tip pointing down. It is elevated so the tip is 1 feet above ground. The cone has height 4 feet and radius 2 feet.

   (a) What is the depth (shown by y in the picture) of the mercury in the tank when exactly half the tank is full, by volume.



h: height of the mercury when the tank is half-full (or half empty if you're a pessimist)

$$\frac{2}{r} = \frac{4}{y} \rightarrow r = \frac{y}{2}$$

$$\frac{1}{2}\int_0^4 \pi \left(\frac{y}{2}\right)^2 dy = \int_0^h \pi \left(\frac{y}{2}\right)^2 dy$$

$$\frac{1}{2} \cdot \frac{\pi}{\cancel{4}} \frac{y^3}{\cancel{3}}\Big|_0^4 = \frac{\pi}{\cancel{4}} \cdot \frac{y^3}{\cancel{3}}\Big|_0^h$$

$$\frac{4^3}{2} = h^3 \longrightarrow h = \frac{4}{\sqrt[3]{2}}$$

   (b) Set up an integral to find the work done in filling this tank to half its capacity with mercury of density 850 pounds per cubic foot pumped from ground level. Do not evaluate this integral.

$$W = \int_0^{4/\sqrt[3]{2}} \pi \left(\frac{y}{2}\right)^2 \cdot 850 \cdot (y+1)\, dy$$