1. (15 points) Evaluate the following integrals. Show all steps. Simplify and box your answers.

(a) $\displaystyle\int \frac{x^3+\sqrt{x}}{x} - \frac{2}{\sqrt{1-x^2}}\,dx$

$$= \int x^2 + x^{-1/2} - 2\,\frac{1}{\sqrt{1-x^2}}\,dx$$

$$= \boxed{\frac{1}{3}x^3 + 2\sqrt{x} - 2\arcsin(x) + C}$$

(b) $\displaystyle\int \sin(t)\cos^3(t) + 2t\,dt$

$$= \underbrace{\int \sin(t)\cos^3(t)\,dt}_{\boxed{\begin{array}{l} u = \cos t \\ du = -\sin t\,dt \end{array}}} + \int 2t\,dt$$

$$= \int u^3(-1)\,dt + t^2 + C$$

$$= -\frac{1}{4}u^4 + t^2 + C$$

$$= \boxed{-\frac{1}{4}\cos^4 t + t^2 + C}$$

(c) $\displaystyle\int_0^{\ln(2)} \frac{e^x}{1-2e^x}\,dx$

$$\boxed{\begin{array}{l} u = 1-2e^x \\ du = -2e^x\,dx \end{array}} \Rightarrow e^x\,dx = -\frac{1}{2}\,du$$

$$= \int_{1-2e^0}^{1-2e^{\ln 2}} \frac{1}{u}\left(-\frac{1}{2}\right)du$$

$$= -\frac{1}{2}\int_{-1}^{-3} \frac{1}{u}\,du = -\frac{1}{2}\ln|u|\Big|_{-1}^{-3}$$

$$= -\frac{1}{2}\ln 3 - \left(-\frac{1}{2}\right)\ln 1$$

$$= \boxed{-\frac{1}{2}\ln 3}$$

2. (10 points) On planet Zorg, the acceleration due to gravity is 8 m/s^2. A Zorgian student throws an orange TI 30X calculator, with some initial velocity v_0, from a cliff 50 meters above the ground. The calculator hits the ground 5 seconds after it was thrown, smashing into pieces.

   (a) (4 points) Compute the initial velocity v_0, and specify if the calculator was thrown up or down.

$a(t) = -8\,m/s^2$ (assuming positive direction is UP)

$v(t) = \int a(t)\,dt = -8t + v_0$

$s(t) = -4t^2 + v_0 t + s_0$

antiderivative

Taking $s(t)$ = position above ground at t seconds: $s(0) = 50$

so: $s(t) = -4t^2 + v_0 t + 50$

$s(5) = 0$: $-4(25) + 5v_0 + 50 = 0$

$5v_0 = 50$

$\boxed{v_0 = 10\,m/s}$

in positive dir. so $\boxed{\text{UP}}$

or:

$\Delta s = -50 = \int_0^5 v(t)\,dt$

$-50 = \left(-8\frac{t^2}{2} + v_0 t\right)\Big|_0^5$

$-50 = -4(25) + v_0(5)$

[...].

   (b) (6 points) Compute the total distance traveled by the ill-fated calculator in the first **3 seconds** after it was thrown.

$v(t) = -8t + 10$, on $[0,3]$ we have:

$v(t) = 0$ at $t = \frac{10}{8} = \frac{5}{4} = 1.25$ sec., $v(t) > 0$ on $[0, \frac{5}{4}]$ &

$v(t) < 0$ on $(5/4, 3]$

Total dist. $= \int_0^3 |v(t)|\,dt$

$= \int_0^{5/4} -8t + 10\,dt + \int_{5/4}^3 8t - 10\,dt$

$= (-4t^2 + 10t)\Big|_0^{5/4} + (4t^2 - 10t)\Big|_{5/4}^3$

$= \left(-\cancel{4}\frac{25}{\cancel{16}_4} + 10\frac{5}{4}\right) + \underbrace{(4(9) - 10(3)) - \left(4\frac{25}{16} - 10\left(\frac{5}{4}\right)\right)}$

$= \underbrace{\frac{25}{4}}_{\text{going up}} + \underbrace{\left(6 + \frac{25}{4}\right)}_{\text{falling down}}$

$= 6 + \frac{25}{2} = \boxed{\frac{37}{2}\,m} = \boxed{18.5\,m}$

3. (10 points) Answer the following three (unrelated) questions:

(a) (4 points) Compute the following limit of a Riemann Sum by first writing it as a definite integral, and then evaluating the integral.

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(\underbrace{\frac{3}{n}}_{\Delta x}\underbrace{\sqrt{4-\frac{3i}{n}}}_{f(x_i)}\right) = \lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{4-x_i}\,\Delta x$$

$\Delta x = \frac{3}{n} \Rightarrow b-a=3$

Let's take $[a,b]=[0,3]$

Then the i^{th} right endpoint is $x_i = a + i\Delta x = 0 + i\frac{3}{n} = \frac{3i}{n}$

$$= \int_0^3 \sqrt{4-x}\,dx$$

$\boxed{u=4-x,\ du=-dx}$

$$= \int_4^1 \sqrt{u}\,(-1)\,du$$

$$= \int_1^4 \sqrt{u}\,du = \frac{2}{3}u^{3/2}\Big|_1^4$$

$$= \frac{2}{3}4^{3/2} - \frac{2}{3} = \frac{2}{3}\cdot 8 - \frac{2}{3} = \boxed{\frac{14}{3}}$$

(b) (4 points) Let $g(x) = \int_0^{2x} \cos(\pi t^2)\,dt$. Compute $g'(1/2)$.

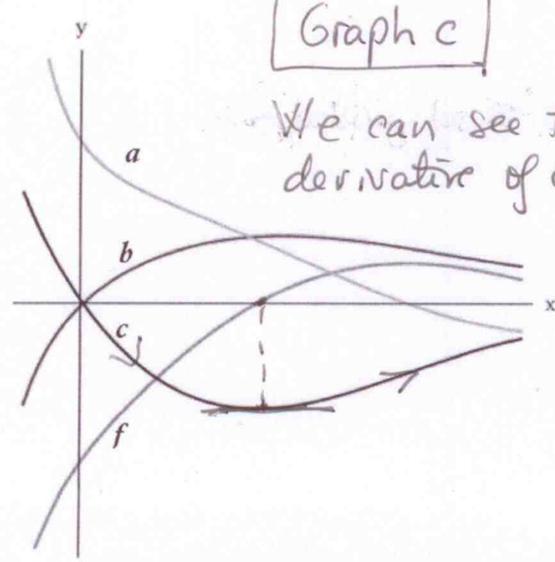
FTC I + Chain Rule: $g'(x) = \cos(\pi(2x)^2)\cdot(2x)'$

$g'(x) = \cos(4\pi x^2)\cdot 2$

Evaluated at $x=\frac{1}{2}$: $g'(\frac{1}{2}) = \cos(4\pi\frac{1}{4})\cdot 2 = \boxed{-2}$

(c) (2 points) The graph of a function f is shown below. Which of the graphs a-c is an antiderivative of f? No need to justify.

$\boxed{\text{Graph c}}$



We can see that f is the derivative of c:

c ↘ when $f<0$

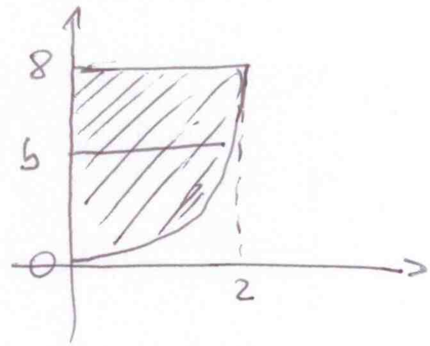
c ↗ when $f>0$

c has a minimum where $f=0$.

$\boxed{x = \sqrt[3]{y}}$

4. (15 points) Consider the region R enclosed by the curve $\boxed{y = x^3}$, the **horizontal** line $y = 8$, and the y-axis.

   (a) (7 points) Find the value of the constant b such that the **horizontal** line $y = b$ divides the region R into two regions of equal area.



Total area:

$$\int_0^8 \sqrt[3]{y}\,dy \quad \left(\text{or: } \int_0^2 (8 - x^3)\,dx\right)$$

$$= \frac{3}{4} y^{4/3} \Big|_0^8 = \frac{3}{4} 8^{4/3} = \frac{3}{4} \cdot 2^4 = \boxed{12}$$

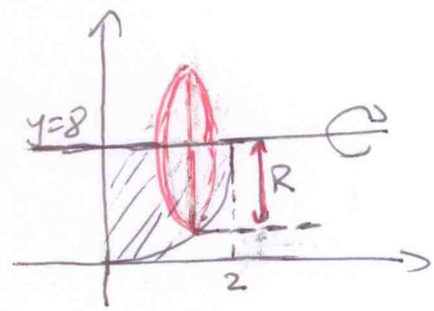
Want: $\int_0^b \sqrt[3]{y}\,dy = \frac{1}{2}(12)$

$$\frac{3}{4} y^{4/3} \Big|_0^b = 6$$

$$\frac{3}{4} b^{4/3} - 0 = 6 \Rightarrow b^{4/3} = \frac{24}{3} = 8$$

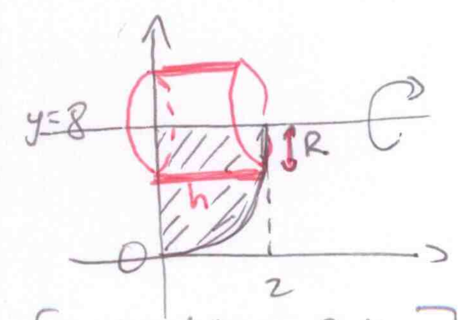
$$\boxed{b = 8^{3/4}}$$

   (b) (8 points) A solid is obtained by rotating the above region R around the **horizontal** line $y = 8$. SET UP integrals equal to the volume of this solid using BOTH the method of disks/washers and the method of cylindrical shells (DO NOT EVALUATE the integrals.)



Disks/Washers:

$$\boxed{V = \int_0^2 \pi (8 - x^3)^2\,dx}$$



Shells:

$$\boxed{V = \int_0^8 2\pi (8 - y)(\sqrt[3]{y})\,dy}$$

[R = top − bottom = $8 - y$

h = right − left = $\sqrt[3]{y} - 0$]