Math 125 L **Midterm 1** **Burdzy, Winter 2007**

Your Name

Your Signature

Student ID #

TA's Name and quiz section (circle):
Andrey Novoseltsev LA Piotr Jagiello LB LC

- Turn off all cell phones, pagers, radios, mp3 players, and other similar devices.
- This exam is closed book. You may use one $8\frac{1}{2}$" $\times$ 11" sheet of handwritten notes (one side).
- Graphing calculators are not allowed.
- Give your answers in exact form, not decimals.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct.
- **Check your work carefully.** We will award only limited partial credit.
- Place [a box around your answer] to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Raise your hand if you have a question.
- This exam has 5 pages, plus this cover sheet. Make sure that your exam is complete.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total	50	

1. (a) (3 points) Compute $\int \left(\frac{x^4 - 2x^7}{x^3} - 4e^x - \sin(x) \right) dx.$

Solution:

$$\int \left(\frac{x^4 - 2x^7}{x^3} - 4e^x - \sin(x) \right) dx = \int \left(x - 2x^4 - 4e^x - \sin(x) \right) dx$$

$$= (1/2)x^2 - (2/5)x^5 - 4e^x + \cos(x) + C$$

(b) (7 points) Compute $\int \frac{e^x}{e^{2x} + 2e^x + 1} \, dx.$

Solution: First, transform the integrand,

$$\int \frac{e^x}{e^{2x} + 2e^x + 1} \, dx = \int \frac{e^x}{(e^x + 1)^2} \, dx.$$

Make the substitution $u = e^x + 1$. Then $du = e^x dx$, and the integral becomes

$$\int \frac{1}{u^2} \, du = -\frac{1}{u} + C.$$

Now substitute back in for u. The answer is

$$\boxed{-\frac{1}{e^x + 1} + C.}$$

2. (a) (4 points) Compute $\int_{-\pi}^{-\pi/2} \left[\cos x - (\cos x)^2\right]^2 \sin(x)\, dx$.

Solution: Substitute $u = \cos(x)$. Then $du = -\sin(x)dx$ and the new limits of integration are $u = -1$ and $u = 0$. We obtain

$$\int_{-\pi}^{-\pi/2} \left((\cos(x) - (\cos(x))^2\right)^2 \sin(x)\, dx = \int_{-1}^{0} (u-u^2)^2(-1)du = -\int_{-1}^{0}(u^2 - 2u^3 + u^4)du$$

$$= -(u^3/3 - 2u^4/4 + u^5/5)|_{u=-1}^{u=0} = -(0 + 1/3 + 1/2 + 1/5) = \boxed{-31/30.}$$

(b) (6 points) The original statement contained a typo: $\int_0^{\sqrt{\pi}} \frac{x\sin(x)}{1+(\cos(x^2))^2}\, dx$.

The problem should have read $\int_0^{\sqrt{\pi}} \frac{x\sin(x^2)}{1+(\cos(x^2))^2}\, dx$.

Solution: Make the substitution $u = \cos(x^2)$. Then $du = -2x\sin(x^2)\,dx$, and $-(1/2)du = x\sin(x^2)\,dx$. The new limits of integration are $u = 1$ and $u = -1$. The integral becomes

$$-\int_1^{-1} \frac{1}{2}\frac{1}{1+u^2}\, du = \int_{-1}^{1} \frac{1}{2}\frac{1}{1+u^2}\, du = \left[\frac{1}{2}\tan^{-1}(u)\right]_{-1}^{1} = \frac{1}{2}(\pi/4 + \pi/4) = \pi/4.$$

3. (10 points) Find the derivative of the function

$$F(x) = \int_{\ln x}^{\cos(x)} \frac{e^t}{1-\tan(t)}\,dt.$$

Solution: For some constant a we let $f(u) = \int_u^a \frac{e^t}{1-\tan(t)}\,dt$ and $g(u) = \int_a^u \frac{e^t}{1-\tan(t)}\,dt$. Then $F(x) = f(\ln x) + g(\cos(x))$. By the Fundamental Theorem of Calculus,

$$f'(u) = -\frac{e^u}{1-\tan(u)}$$

and

$$g'(u) = \frac{e^u}{1-\tan(u)}$$

By the chain rule,

$$\frac{d}{dx}F(x) = f' \cdot \frac{d}{dx}(\ln x) + g' \cdot \frac{d}{dx}\cos(x) = -\frac{e^{\ln x}}{1-\tan(\ln x)} \cdot \frac{1}{x} + \frac{e^{\cos(x)}}{1-\tan(\cos(x))} \cdot (-\sin(x))$$

$$= \boxed{-\frac{1}{1-\tan(\ln x)} - \frac{\sin(x)e^{\cos(x)}}{1-\tan(\cos(x))}}.$$

4. The mileage of a car depends on the road conditions (freeway, unpaved road, mountains). Let $f(x)$ denote the mileage measured in miles per gallon, where x is the distance from the starting point of the journey. Let $g(x)$ be the current rate of consumption of gas by a car, measured in gallons per mile, where x is, as before, the distance from the starting point of a journey.

   (a) (1 point) Find the relationship between $f(x)$ and $g(x)$.

   Solution: $\boxed{g(x) = 1/f(x)}$

   (b) (2 points) Write a formula in terms of $g(x)$ for the total amount of gas used if the journey was 143 miles.

   Solution: $\boxed{\int_0^{143} g(x)dx}$

   (c) (2 points) Write a formula in terms of $f(x)$ for the total amount of gas used if the journey was 211 miles.

   Solution: $\boxed{\int_0^{211} \frac{1}{f(x)}dx}$

   (d) (5 points) The "Edsel" model of Ford had low mileage. A brand new car had 20 MPG (miles per gallon) mileage but the mileage declined steadily over the lifetime of the car to 15 MPG. Assume that the lifetime of the car was 100K (100 thousand miles) and the mileage was a linear function of miles driven. Find the total amount of gas used by Edsel over its lifetime.

   Solution: Let x denote the number of miles driven and let $f(x)$ stand for the mileage. Note that x varies from 0 to 100,000 over the lifetime of a car. The function $f(x)$ is linear with $f(0) = 20$ and $f(100{,}000) = 15$ so $f(x) = -\frac{1}{20{,}000}x + 20$. The number of gallons of gas burnt per mile is $g(x) = 1/f(x)$. The total number of gallons of gas used over the lifetime of Edsel was equal to

   $$\int_0^{100{,}000} g(x)\,dx = \int_0^{100{,}000} \frac{1}{f(x)}\,dx = \int_0^{100{,}000} \frac{1}{-\frac{1}{20{,}000}x + 20}\,dx.$$

   We make the substitution $u = -\frac{1}{20{,}000}x + 20$, so $du = -\frac{1}{20{,}000}\,dx$, and $-20{,}000\,du = dx$. The new limits of integration are $u = 20$ and $u = 15$. We obtain

   $$\int_0^{100{,}000} \frac{1}{-\frac{1}{20{,}000}x + 20}\,dx = \int_{20}^{15} \frac{1}{u} \cdot (-20{,}000)du = (-20{,}000 \ln u)_{u=20}^{u=15} = -20{,}000(\ln 15 - \ln 20)$$

   $$= \boxed{20{,}000 \ln(4/3)} \approx 5{,}753.64.$$

5. A ball is dropped from a window located 64 ft above the ground. Assume that the gravitational acceleration of the ball is $-32\,\mathrm{ft/sec}^2$.

   (a) (4 points) Suppose that the ball is released with no initial vertical velocity. Find the time of the impact and the vertical velocity of the ball at the time of impact.

   Solution: Let t stand for time, $d(t)$ for the distance to the ground, $v(t)$ for velocity, and $a(t)$ for the acceleration of the ball. We have $a(t) = -32$, $v'(t) = a(t)$, and $d'(t) = v(t)$. The initial conditions are $d(0) = 64$ and $v(0) = 0$. Since $v'(t) = -32$, we have $v(t) = -32t + C_1$. The initial velocity is $v(0) = 0$ so $C_1 = 0$ and $v(t) = -32t$. We note that $d'(t) = v(t) = -32t$, so $d(t) = -16t^2 + C_2$. We must have $d(0) = 64$ so $d(t) = -16t^2 + 64$. At the time of impact t_0 we have $d(t_0) = 0$, so $-16t_0^2 + 64 = 0$. We conclude that $\boxed{t_0 = 2}$. The velocity at the time of impact is $v(t_0) = -32t_0 = -32 \cdot 2 = \boxed{-64}$.

   (b) (6 points) What initial vertical velocity should be given to the ball so that the ball hits the ground in half the time computed in the first part of the problem?

   Solution: We follow the same argument as in the first part, except that the initial velocity $v(0) = v_0$, which clearly cannot be equal to 0.

   Recall that t denotes time, $d(t)$ denotes the distance to the ground, $v(t)$ is velocity, and $a(t)$ is the acceleration of the ball. We have $a(t) = -32$, $v'(t) = a(t)$, and $d'(t) = v(t)$. The initial conditions are $d(0) = 64$ and $v(0) = v_0$. Since $v'(t) = -32$, we have $v(t) = -32t + C_1$. The initial velocity is $v(0) = v_0$ so $C_1 = v_0$ and $v(t) = -32t + v_0$. We note that $d'(t) = v(t) = -32t + v_0$, so $d(t) = -16t^2 + v_0 t + C_2$. We must have $d(0) = 64$ so $d(t) = -16t^2 + v_0 t + 64$. At the time of impact t_0 we have $d(t_0) = 0$, so $-16t_0^2 + v_0 t_0 + 64 = 0$. We conclude that $t_0 = \dfrac{-v_0 \pm \sqrt{v_0^2 + 4 \cdot 16 \cdot 64}}{-32}$. The positive solution is $t_0 = \dfrac{v_0 + \sqrt{v_0^2 + 4096}}{32}$. This is supposed to be equal to one half of 2 seconds, that is, 1 second. We have to solve the equation

$$
\begin{aligned}
1 &= \frac{v_0 + \sqrt{v_0^2 + 4096}}{32} \\
32 &= v_0 + \sqrt{v_0^2 + 4096} \\
32 - v_0 &= \sqrt{v_0^2 + 4096} \\
1024 - 64v_0 + v_0^2 &= v_0^2 + 4096 \\
v_0 &= \frac{4096 - 1024}{-64} = \boxed{-48}\ \mathrm{ft/sec}.
\end{aligned}
$$

   Note that the initial velocity has to be negative because the ball has to be thrown towards the ground so that it hits the ground sooner than a ball released with no initial velocity.