1. **[12 points]** Evaluate the following integrals. Show all steps. Simplify and box your answer.

(a) **[4 points]** $\int \frac{1+x^2}{x^3}\,dx$

$$= \int\left(\frac{1}{x^3} + \frac{x^2}{x^3}\right)dx = \int\left(x^{-3} + \frac{1}{x}\right)dx$$

$$= \frac{x^{-2}}{-2} + \ln|x| + C$$

$$= \boxed{-\frac{1}{2x^2} + \ln|x| + C}$$

(b) **[4 points]** $\int \frac{x^3}{1+x^2}\,dx$

$$\boxed{u = 1+x^2, \quad du = 2x\,dx}$$

$$x^2 = u-1 \qquad x\,dx = \frac{1}{2}du$$

$$= \int \frac{u-1}{u}\,\frac{1}{2}du$$

$$= \frac{1}{2}\int 1 - \frac{1}{u}\,du = \frac{1}{2}(u - \ln|u|) + C$$

$$= \boxed{\frac{1}{2}(1+x^2) - \frac{1}{2}\ln(1+x^2) + C}$$

(c) **[4 points]** $\int_0^2 y^2 e^{-y^3}\,dy$

$$\boxed{u = -y^3 \quad du = -3y^2\,dy}$$

$$y^2\,dy = -\frac{1}{3}du$$

$$y=0 \Rightarrow u = -(0)^3 = 0$$

$$y=2 \Rightarrow u = -(2)^3 = -8.$$

$$= \int_0^{-8} e^u\left(-\frac{1}{3}\right)du$$

$$= \frac{1}{3}\int_{-8}^{0} e^u\,du$$

$$= \frac{1}{3}e^u\Big|_{-8}^{0}$$

$$= \frac{1}{3}e^0 - \frac{1}{3}e^{-8} = \boxed{\frac{1}{3} - \frac{1}{3}e^{-8}}.$$

2. [**10 points**]

(a) [**5 points**] Let $F(x) = \int_1^{\sqrt{x}} \arctan(t)\, dt$. Evaluate $F(1)$ and $F'(1)$.

$F(1) = \int_1^{\sqrt{1}} \arctan(t)\, dt = \boxed{0}$ (same upper and lower bound!)

$F'(x) = \arctan(\sqrt{x}) \cdot (\sqrt{x})'$ ⟸ FTC I + Chain Rule

$= \arctan(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}$

$F'(1) = \arctan(1) \cdot \frac{1}{2\sqrt{1}} = \boxed{\frac{\pi}{8}}$

(b) [**5 points**] Compute $\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{1}{\left(1+\frac{9i}{n}\right)^2} \frac{9}{n} \right)$ by writing it as a definite integral and then solving the integral.

$\Delta x = \frac{b-a}{n} = \frac{9}{n} \Rightarrow b-a = 9$.

One way: If we choose $[a,b] = [0,9]$, then $x_i = 0 + i\Delta x = \frac{9i}{n}$

so $\frac{1}{(1+\frac{9i}{n})^2} = \frac{1}{(1+x_i)^2} = f(x_i) \Rightarrow f(x) = \frac{1}{(1+x)^2}$

Then the limit $= \boxed{\int_0^9 \frac{1}{(1+x)^2} dx} \underset{\substack{u=1+x \\ du=dx}}{=} \int_1^{10} \frac{1}{u^2} du = -\frac{1}{u}\Big|_1^{10}$

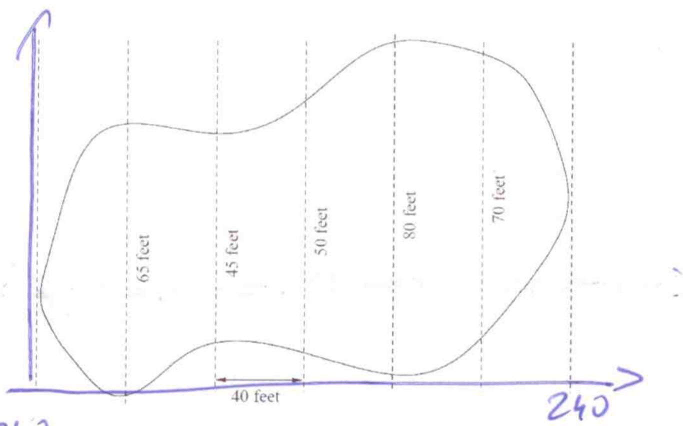
$= -\frac{1}{10} + 1 = \boxed{\frac{9}{10}}$

Another way: If we choose $[a,b] = [1,9]$, then $x_i = 1 + i\Delta x = 1 + \frac{9i}{n}$

so $\frac{1}{(1+\frac{9i}{n})^2} = \frac{1}{x_i^2} \Rightarrow f(x) = \frac{1}{x^2}$

Then the limit $= \boxed{\int_1^{10} \frac{1}{x^2} dx} = -\frac{1}{x}\Big|_1^{10} = \boxed{\frac{9}{10}}$

3. **[8 points]** The widths across a lake were measured at 40 ft intervals.



$A = \int_0^{240} W(x)\,dx$ $\quad$ $W(x) =$ width of lake at x

(a) Use R_6 to estimate the area of the lake (right endpoints, with $n = 6$ subintervals)

$R_6 = \sum_{i=1}^{6} W(x_i)\,\Delta x$ where $\Delta x = 40$ ft & $W(x_1) = 65, \ldots$

$= (65 + 45 + 50 + 80 + 70 + 0)\,40$

$= (310)(40) = \boxed{12{,}400 \text{ ft}^2}$

(b) Use the midpoint rule with the given data to give another estimate of the lake area.

Use $n = 3$ so every other value is at a midpoint:

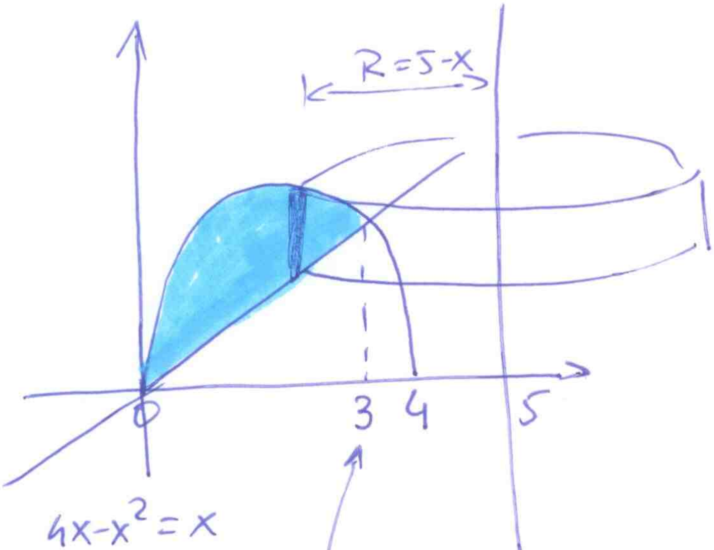
$M_3 = \underbrace{W(m_1)}\,\underbrace{\Delta x} + W(m_2)\,\Delta x + W(m_3)\,\Delta x$ $\quad$ with $\Delta x = \frac{240}{3} = 80$

$= (65)(80) + (50)(80) + (70)(80)$

$= \boxed{14{,}800 \text{ ft}^2}$

4. **[10 points]** For each of the following, set up a definite integral that represents the volume described in the problem. DO NOT SIMPLIFY OR EVALUATE the integrals.

(a) **[6 points]** Let R be the region enclosed between the graphs of $f(x) = 4x - x^2$ and $g(x) = x$. The region R is rotated about the vertical line $x = 5$ to form a solid of revolution.



Given the shape of the region, it is easiest to integrate in x, so use SHELLS:

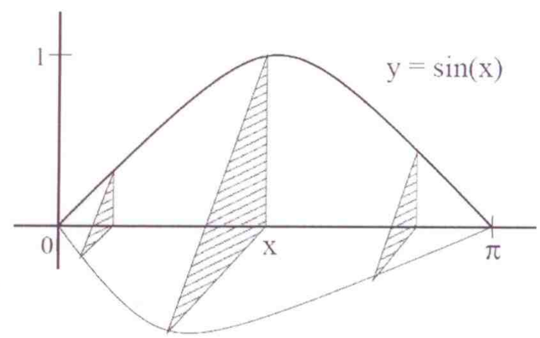
$$\boxed{V = \int_0^3 2\pi \underbrace{(5-x)}_{\text{radius of shell}} \underbrace{(4x - x^2 - x)}_{\text{height of shell}}\,dx}$$

$4x - x^2 = x$

$x^2 - 3x = 0$

$x = 0$ & $\boxed{x = 3}$

(b) **[4 points]** Let S be the solid shown below. Each cross section of S by a plane perpendicular to the x-axis for $0 \le x \le \pi$ is an isosceles right triangle, whose base equals its height (see the figure).



isoceles right triangle cross sections

$$V = \int_0^{\pi} A(x)\,dx$$

Note: this is not a solid of revolution so neither disks/washers, nor shells apply.

$A(x)$ = area of the isosceles right triangle at x

$$= \frac{1}{2}\underbrace{(\sin x)}_{\text{base}}\underbrace{(\sin x)}_{\text{height}} = \frac{1}{2}\sin^2 x$$

$$\boxed{V = \int_0^{\pi} \frac{1}{2}\sin^2 x\,dx}$$

5. **[10 points]** Pete is driving his car along a straight and very busy street. He starts at his work place and needs to deliver a package to a customer. Being distracted, he starts going in the wrong direction, but realizes his mistake after a while. The velocity of his car at t hours during his trip is given by $v(t) = 120t^2 - 70t$, in miles per hour.

(a) **[4 points]** Pete reaches his customer's office after one hour. How far away is the customer's office from Pete's work place?

This question asks for the displacement of Pete's car in 1 hour, so:

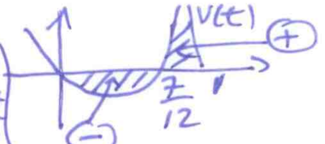
$$\int_0^1 v(t)\,dt = \int_0^1 120t^2 - 70t\,dt$$
$$= (40t^3 - 35t^2)\Big|_0^1 = (40-35) - (0-0)$$
$$= \boxed{5 \text{ miles}}$$

(b) **[6 points]** Pete's car gets 20 miles per gallon of fuel. How much fuel did Pete use up for this journey? Round your answer to two decimal digits.

Here we first need the total distance so

$$d = \int_0^1 |v(t)|\,dt$$

$120t^2 - 70t = 0$ at $t = 0$ & $t = \frac{7}{12}$



$$= \int_0^{7/12} -(120t^2 - 70t)\,dt + \int_{7/12}^1 (120t^2 - 70t)\,dt$$
$$= \int_0^{7/12} (70t - 120t^2)\,dt + \int_{7/12}^1 (120t^2 - 70t)\,dt$$
$$= (35t^2 - 40t^3)\Big|_0^{7/12} + (40t^3 - 35t^2)\Big|_{7/12}^1$$
$$= \frac{1715}{432} + \frac{3875}{432} \quad (\text{OR} \cong 3.9699 + 8.9699$$
$$= \frac{2795}{216} \cong 12.94 \text{ miles}$$

Fuel used $\cong \dfrac{12.94 \text{ miles}}{20 \text{ miles/gallon}} = 0.647 \cong$ 0.65 gallons