University of Washington **Department of Mathematics**
Midterm Exam #1 - Solutions **Math 125 A & B** **October 22, 2015**

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1. (a) (5 points) Let $u = 2v^3$ so that $du = 6v^2\,dv$. When $v = 0$ we also have $v = 0$ but when $v = 1$ we have $u = 2$, so we obtain via substitution

$$\int_0^1 v^2\cos(2v^3)dv = \frac{1}{6}\int_0^2 \cos(u)du = \frac{1}{6}\sin(u)\Big|_0^2 = \boxed{\frac{1}{6}\sin(2).}$$

(b) (5 points) Let $u = \ln(\cos\theta)$ so that, using the chain rule, we have

$$du = \frac{1}{\cos\theta}\frac{d}{d\theta}(\cos\theta)\,d\theta = -\frac{\sin\theta}{\cos\theta}\,d\theta = -\tan\theta\,d\theta.$$

Therefore

$$\int \tan\theta\,\ln(\cos\theta)\,d\theta = -\int u\,du = -\frac{u^2}{2} + C = \boxed{-\frac{(\ln(\cos\theta))^2}{2} + C}$$

2. (10 points) If $f(x) = \int_0^{\sin x}\sqrt{1+t^2}dt$ and $g(y) = \int_3^y f(x)dx$, find $g''(\pi/6)$.
First observe that by Part 1 of the Fundamental Theorem of Calculus we have $g'(y) = f(y)$ which implies that $g''(y) = f'(y)$ and in particular that

$$\boxed{g''(\pi/6) = f'(\pi/6).}$$

Again applying Part 1 of the Fundamental Theorem of Calculus, this time togther with the chain rule, we see that $f'(x) = \sqrt{1+\sin^2(x)}\,\cos(x)$ so $f'(\pi/6) = \cos(\pi/6)\sqrt{1+\sin^2(\pi/6)}$. Since $\cos(\pi/6) = \sqrt{3}/2$ and $\sin(\pi/6) = 1/2$ this gives us (after simplifying) that $\boxed{g''(\pi/6) = \frac{\sqrt{15}}{4}}$.

3. (10 points) An object is moving on the x-axis with acceleration $a(t) = 2t + 4$ for $0 \le t \le 4$. Its velocity at time $t = 0$ is $v(0) = -5$.
(a) (5 points) What is the displacement of the object between times $t = 0$ and $t = 4$?
First note that since the velocity is the antiderivative of the acceleration we see that the velocity is given by $v(t) = t^2 + 4t + v_0$. Moreover we have $v_0 = v(0) = -5$, therefore the velocity is

$$v(t) = t^2 + 4t - 5$$

The displacement is simply the definite integral of the velocity over the indicated time interval, so this is given by

$$\int_0^4 v(t)dt = \int_0^4 (t^2 + 4t - 5)dt = \left[\frac{t^3}{3} + 2t^2 - 5t\right]\Bigg|_0^4 = \frac{64}{3} + 32 - 20 = \boxed{\frac{100}{3}}.$$

(b) (5 points) What is the total distance travelled by the object between times $t = 0$ and $t = 4$?

The total distance travelled is given by the definite integral of the **absolute value** of the velocity function over the given time interval. In order to evaluate this we need to figure out where the velocity is positive and where it is negative. Setting $v(t) = t^2 + 4t - 5 = 0$ we can factor to see that the solutions are $t = -5$ and $t = 1$. So between $t = 0$ and $t = 4$ the only place that the velocity can change sign is at $t = 1$. Next we observe (by checking particular values for example) that $v(t) < 0$ on (01) and $v(t) > 0$ on $(1, 4)$. Therefore the total distance travelled is given by

$$\begin{aligned}
\int_0^4 |v(t)|dt &= \int_0^1 -(v(t))dt + \int_1^4 v(t)dt = \int_0^1 -(t^2+4t-5)dt + \int_1^4 (t^2+4t-5)dt \\
&= -\left[\frac{t^3}{3} + 2t^2 - 5t\right]\Bigg|_0^1 + \left[\frac{t^3}{3} + 2t^2 - 5t\right]\Bigg|_1^4 \\
&= -\left[\frac{1}{3} + 2 - 5\right] + 0 + \frac{100}{3} - \left[\frac{1}{3} + 2 - 5\right] = \frac{8}{3} + \frac{100}{3} + \frac{8}{3} = \boxed{\frac{116}{3}}.
\end{aligned}$$

4. (10 points) Find the area of the region bounded by the curves

$$y = \frac{1}{1+x^2} \quad \text{and} \quad y = \frac{1}{2}.$$

Note that the curves intersect at the points $(-1, \frac{1}{2})$ and $(1, \frac{1}{2})$ and they bound the region in between. In this region $\frac{1}{1+x^2} \geq \frac{1}{2}$. Thus the area, A, of the region bounded by the 2 curves is

$$\begin{aligned}
A = \int_{-1}^1 \left[\frac{1}{1+x^2} - \frac{1}{2}\right] dt &= \arctan x - \frac{x}{2}\Big|_{-1}^1 \\
&= \arctan(1) - \arctan(-1) - 1 = \boxed{\frac{\pi}{2} - 1}
\end{aligned}$$

5. (10 points) Consider the region R bounded between the parabola $y = x^2$ and the line $y = 3x$.

(a) (2 points) Sketch the region. This is easily done since one is a line through the origin and the other is a parabola. The points of intersection are $(0, 0)$ and $(3, 9)$.

(b) (4 points) Find the volume of the solid obtained by rotating the region R about the y-axis.

(This can be done by either washers or shells. We will provide both solutions, you only needed to provide one of these.)

Cylindrical Shells: Since the axis of rotation is vertical the integral is in the x-variable. The vertical line segments which when rotated about the axis produce the shells start at $x = 0$ and finish at $x = 3$, the radius is x and the height is $3x - x^2$. Thus the volume is given by

$$\begin{aligned} V &= \int_0^3 2\pi x(3x - x^2)\, dx = 2\pi \int_0^3 (3x^2 - x^3)\, dx \\ &= 2\pi \left[x^3 - \frac{x^4}{4} \right]\Bigg|_0^3 = 2\pi \left[27 - \frac{81}{4} \right] = \boxed{\frac{27}{2}\pi}. \end{aligned}$$

Washers: Since the axis of rotation is vertical the integral is in the y-variable. The slices perpendicular to the axis yielding washers start at $y = 0$ and finish at $y = 9$. The outer radius is $R(y) = \sqrt{y}$ and the inner radius is $r(y) = y/3$. Thus the volume is given by

$$\begin{aligned} V &= \int_0^9 \left[\pi(\sqrt{y})^2 - \pi(\frac{y}{3})^2 \right] dy = \pi \int_0^9 \left[y - \frac{y^2}{9} \right] dy \\ &= \pi \left[\frac{y^2}{2} - \frac{y^3}{27} \right]\Bigg|_0^9 = \pi \left[\frac{81}{2} - \frac{81}{3} \right] = \boxed{\frac{27}{2}\pi}. \end{aligned}$$

(c) (4points) Set up a definite integral for the volume obtained by rotating the region R about the horizontal line $y = 10$. You may either use the method of cylindrical shells, or the method of washers. Indicate your answer in one of the appropriate places below. If you write both integrals we will grade each out of 2 points. If you only write one we will grade it out of 4 points. **DO NOT EVALUATE EITHER INTEGRAL.**

Cylindrical Shells: Since the axis of rotation is horizontal the integral is in the y-variable. The horizontal line segments which when rotated about the axis produce the shells start at $y = 0$ and finish at $y = 9$, the radius is the distance to the axis which is $10 - y$ and the height is $\sqrt{y} - \frac{y}{3}$. Thus the volume is given by

$$\boxed{V = \int_0^9 2\pi(10 - y)(\sqrt{y} - \frac{y}{3})\, dy.}$$

Washers: Since the axis of rotation is horizontal the integral is in the x-variable. The slices perpendicular to the axis yielding washers start at $x = 0$ and finish at $x = 3$. The outer radius is $R(x) = 10 - x^2$ and the inner radius is $r(x) = 10 - 3x$. Thus the volume is given by

$$\boxed{V = \int_0^3 \left[\pi(10 - x^2)^2 - \pi(10 - 3x)^2 \right] dx.}$$