Final solutions

1. (12 points) Evaluate the following integrals. Simplify your answers. Show clear and complete work.

(a) (6 points) $\int_0^1 x\ln(x^2+1)\,dx = \frac{1}{2}\int_1^2 \ln(u)\,du$

① $\boxed{\begin{array}{l} u = x^2+1 \\ du = 2x\,dx \end{array}}$

$= \frac{1}{2}\left[u\ln u\Big|_1^2 - \int_1^2 \frac{1}{\cancel{u}}\cancel{u}\,du\right]$

② IBP: $\boxed{\begin{array}{ll} w = \ln u & dv = du \\ dw = \frac{1}{u}du & v = u \end{array}}$

$= \frac{1}{2}\left[u\ln u - u\right]\Big|_1^2$

$= \frac{1}{2}\left[(2\ln 2 - 2) - (1\cancel{\ln 1}^{0} - 1)\right]$

$= \frac{1}{2}\left[2\ln 2 - 1\right] = \ln 2 - \frac{1}{2}$

Answer: $\ln(2) - 1/2$

(b) (6 points) $\int_{-\pi/2}^{\pi/2} \sin^2(x)\cos^3(x)\,dx$

by symmetry (even fct)

$= 2\int_0^{\pi/2} \sin^2(x)\cos^2(x)\cos(x)\,dx$

$= 2\int_0^{\pi/2} \sin^2(x)(1-\sin^2 x)\cos x\,dx$ $\quad u = \sin x$, $du = \cos x\,dx$

$= 2\int_0^1 u^2(1-u^2)\,du$

$= 2\int_0^1 u^2 - u^4\,du$

$= 2\left[\frac{1}{3}u^3 - \frac{1}{5}u^5\right]\Big|_0^1$

$= 2\left[\left(\frac{1}{3}-\frac{1}{5}\right) - 0\right] = \frac{4}{15}$

Answer: $4/15$

2. (14 points) Evaluate the following integrals. Simplify your answers. Show clear and complete work.

(a) (7 points) $\displaystyle\int \frac{x+1}{9-x^2}\,dx$

$$= \int \frac{(2/3)}{3-x} + \frac{(-1/3)}{3+x}\,dx$$

$$= -\frac{2}{3}\ln|3-x| - \frac{1}{3}\ln|3+x| + C$$

Partial Fractions:

$$\frac{x+1}{(3-x)(3+x)} = \frac{A}{3-x} + \frac{B}{3+x}$$

$$x+1 = A(3+x)+B(3-x)$$

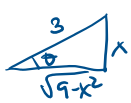
$x=3$: $4 = 6A \Rightarrow A = 2/3$

$x=-3$: $-2 = 6B \Rightarrow B = -1/3$

Answer: $-2/3 \ln|3-x| - 1/3 \ln|3+x| + C$

(b) (7 points) $\displaystyle\int \frac{x^2}{\sqrt{9-x^2}}\,dx = \int \frac{9\sin^2\theta}{\sqrt{9-9\sin^2\theta}}\,3\cos\theta\,d\theta$

Trig Sub: $x = 3\sin\theta$
$dx = 3\cos\theta\,d\theta$



$$= \int \frac{9\sin^2\theta}{\cancel{3\cos\theta}}\,\cancel{3\cos\theta}\,d\theta$$

$$= 9\int \frac{1-\cos(2\theta)}{2}\,d\theta$$

$$= \frac{9}{2}\left[\theta - \frac{\sin 2\theta}{2}\right] + C$$

$$= \frac{9}{2}\left[\theta - \sin\theta\cos\theta\right] + C$$

$$= \frac{9}{2}\left[\arcsin\left(\frac{x}{3}\right) - \frac{x}{3}\,\frac{\sqrt{9-x^2}}{3}\right] + C$$

Answer: $\frac{9}{2}\arcsin\left(\frac{x}{3}\right) - \frac{1}{2}x\sqrt{9-x^2} + C$

(simplify your final answer so no inverse trig appears inside a trig function)

3. (10 points) Determine whether the following improper integrals are convergent or divergent. If the integral is convergent, evaluate it. If it is divergent, fully indicate why. For credit, you must show all appropriate limits and computations.

(a) (4 points) $\int_0^4 \frac{2}{x\sqrt{x}}\,dx$ $= \lim_{t\to 0^+}\left(\int_t^4 \frac{2}{x\sqrt{x}}dx\right)$

$$= \lim_{t\to 0^+}\left(2\int_t^4 x^{-3/2}dx\right)$$

$$= \lim_{t\to 0^+}\left(2\,(-2)\left[\frac{1}{\sqrt{x}}\right]\Big|_t^4\right)$$

$$= (-4)\lim_{t\to 0^+}\left[\frac{1}{2}-\frac{1}{\sqrt{t}}\right]$$

$$= (-4)\left[\tfrac{1}{2}-\infty\right] = +\infty$$

Answer: DIVERGENT

(b) (6 points) $\int_3^\infty \frac{1}{x^2+x}\,dx$

PF: $\frac{1}{x(x+1)} = \frac{A}{x}+\frac{B}{x+1}$

$1 = A(x+1)+Bx$

$\Rightarrow A=1$

& $B=-1$.

$$= \lim_{t\to\infty}\left(\int_3^t \frac{1}{x(x+1)}\,dx\right)$$

$$= \lim_{t\to\infty}\left(\int_3^t \frac{1}{x}+\frac{(-1)}{x+1}\,dx\right)$$

$$= \lim_{t\to\infty}\left[\ln|x| - \ln|x+1|\right]\Big|_3^t$$

$$= \lim_{t\to\infty}\left[\ln\left|\frac{t}{t+1}\right| - \ln\left(\frac{3}{4}\right)\right]$$

$$= \ln\left|\lim_{t\to\infty}\frac{1}{1+1/t}\right| - \ln(3/4)$$

$$= \ln(1)-\ln(3/4)$$

$$= -\ln(3/4) = \ln(4/3)$$

Answer: $\ln(4/3) = -\ln(3/4)$ $\cong 0.2877$

4. (10 points) Ziggy the cat is walking back and forth along a straight line represented by the x-axis, and her position in meters along this path at t seconds after 12:00 noon is given by $x = s(t)$.

Suppose that her initial position at 12:00 is $s(0) = 0$, and that Ziggy starts out walking in the positive (eastward) direction, but at 12:01 she is west of her starting point.

Match each mathematical expression in the left column below to the letter (A)–(H) for the correct description of what it represents in the right column below. Write "X" if the expression does not match any of (A)–(H). Assume all units in (A)–(H) are in meters or meters per second, as appropriate. You do not need to justify your answers.

Note: Any particular letter (A)–(H) or (X) may appear once, more than once, or not at all.

Match each of these expressions:

1. $|s(60)| =$ H
2. $s'(60) =$ F
3. $|s'(60)| =$ G
4. $\dfrac{s(60)-s(0)}{60-0} =$ D
5. $\displaystyle\int_0^{60} s(t)\,dt =$ X
6. $\displaystyle\int_0^{60} s'(t)\,dt =$ B
7. $\displaystyle\int_0^{60} |s'(t)|\,dt =$ C
8. $\displaystyle\frac{1}{60}\int_0^{60} s'(t)\,dt =$ D
9. $\displaystyle\lim_{h\to 0}\frac{s(60+h)-s(60)}{h} =$ F
10. $\displaystyle\frac{d}{dt}\left(\int_0^t s(x)\,dx\right)$ at $t = 60$ A

...To what it represents among these options:

(A) Ziggy's position at 12:01.

(B) Ziggy's displacement from 12:00 to 12:01.

(C) The total distance Ziggy traveled from 12:00 to 12:01.

(D) Ziggy's average velocity from 12:00 to 12:01.

(E) Ziggy's average speed from 12:00 to 12:01.

(F) Ziggy's instantaneous velocity at 12:01.

(G) Ziggy's instantaneous speed at 12:01.

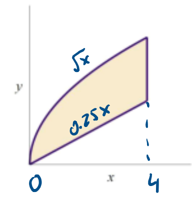
(H) The distance from Ziggy's starting point to her position at 12:01.

(X) None of the descriptions (A)-(H) above match

5. The region R shown below is bounded at the top by $y = \sqrt{x}$, at the bottom by $y = 0.25x$, and on the right by $x = 4$.

(a) (3 points) Set up integral(s) with respect to the variable x to find the **area** of this region R. Then evaluate the integral(s) and compute the area.

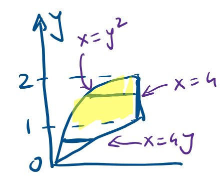
$$A = \int_0^4 (\sqrt{x} - 0.25x)\,dx = \left(\frac{2}{3}x^{3/2} - \frac{0.25}{2}x^2\right)\Big|_0^4$$
$$= \left(\frac{2}{3}(\sqrt{4})^3 - \frac{0.25}{2}(4)^2\right) - (0)$$
$$= \frac{16}{3} - 2$$
$$= \frac{16-6}{3}$$



Answer: Integral(s): $\int_0^4 (\sqrt{x} - 0.25x)\,dx$, Area = $10/3$

(b) (4 points) Set up integral(s) with respect to the variable y to find the **area** of the region R. Then evaluate the integral(s) to compute the area. Check that your answer matches (a).

$$A = \int_0^1 (4y - y^2)\,dy + \int_1^2 (4 - y^2)\,dy$$
$$= \left(2y^2 - \tfrac{1}{3}y^3\right)\Big|_0^1 + \left(4y - \tfrac{1}{3}y^3\right)\Big|_1^2$$
$$= \left(2 - \cancel{\tfrac{1}{3}}\right) - (0) + \left(8 - \tfrac{8}{3}\right) - \left(4 - \cancel{\tfrac{1}{3}}\right)$$
$$= 6 - 8/3 = 10/3$$



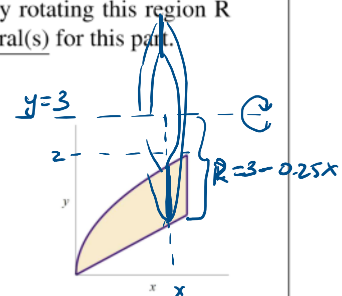
Answer: Integral(s): $\int_0^1 (4y - y^2)\,dy + \int_1^2 (4 - y^2)\,dy$, Area = $10/3$

(c) (5 points) Set up integral(s) to find the **volume** of the solid obtained by rotating this region R about the horizontal line $y = 3$. Do NOT compute or simplify your integral(s) for this part.

Washers in x: $\int_0^4 \pi\,(3 - 0.25x)^2 - \pi\,(3 - \sqrt{x})^2\,dx$

OR

Shells in y: $\int_0^1 2\pi\,\underbrace{(3-y)}_{\text{radius}}\underbrace{(4y - y^2)}_{\text{height}}\,dy + \int_1^2 2\pi\,(3-y)(4-y^2)\,dy$



Answer: Integral(s): $\int_0^4 \pi\,(3 - 0.25x)^2 - \pi\,(3 - \sqrt{x})^2\,dx$

6. (a) (6 points) Compute the exact arc length of the curve $y = \frac{1}{3}x^{3/2}$ from $x = 0$ to $x = 12$.

$$L = \int_0^{12} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \longrightarrow y = \frac{1}{3}x^{3/2} \Rightarrow \frac{dy}{dx} = \frac{1}{\cancel{3}}\frac{\cancel{3}}{2}x^{1/2} = \frac{1}{2}\sqrt{x}.$$

$$= \int_0^{12} \sqrt{1+\frac{1}{4}x}\; dx$$

$$= \frac{1}{2}\int_0^{12} \sqrt{4+x}\; dx \longrightarrow \boxed{u = 4+x,\ du = dx}$$

$$= \frac{1}{2}\int_4^{16} \sqrt{u}\, du$$

$$= \frac{1}{\cancel{2}}\frac{\cancel{2}}{3}u^{3/2}\Big|_4^{16} = \frac{1}{3}(64-8) = 56/3$$

Answer: 56/3

(b) (4 points) Use the Trapezoidal Rule with $n = 4$ sub-intervals to approximate the arc length from part (a). Show all steps, and round your final answer to the nearest 4 decimal digits.

$$L = \int_0^{12} \sqrt{1+\frac{1}{4}x}\; dx \qquad \Delta x = 12/4 = 3, \quad f(x) = \sqrt{1+x/4}$$

$$T_4 = \frac{\Delta x}{2}\left[f(0) + 2f(3) + 2f(6) + 2f(9) + f(12)\right].$$

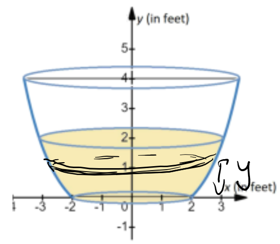
$$= \frac{3}{2}\left[\sqrt{1+0} + 2\sqrt{1+3/4} + 2\sqrt{1+6/4} + 2\sqrt{1+9/4} + \sqrt{1+12/4}\right].$$

$$= \frac{3}{2}\left(1 + \cancel{2}\frac{\sqrt{7}}{\cancel{2}} + \cancel{2}\frac{\sqrt{10}}{\cancel{2}} + \cancel{2}\frac{\sqrt{13}}{\cancel{2}} + 2\right)$$

$$= \frac{3}{2}\left(3 + \sqrt{7} + \sqrt{10} + \sqrt{13}\right) \cong 18.62037\ldots$$

Answer: 18.6204

7. (8 points) A container is created by rotating the portion of the curve $x = \ln(y+1)+2$ with $0 \le y \le 4$ feet around the y-axis. The container is filled to a height of 2 feet with paraffin oil, which weighs 49.9 lbs per cubic foot.



(a) (2 points) Consider a very thin layer of oil, at coordinate y and of thickness Δy. Write an expression in y and Δy that approximately equals the **weight, in lbs**, of just this one thin horizontal layer of oil. There should be no integral in your answer.

$R = x = \ln(y+1)+2$

$\to$ weight = (volume of disk layer in $\cancel{ft^3}$)(49.9 lbs/$\cancel{ft^3}$) lbs.

$= (\pi R^2 \Delta y)\, 49.9$

with

$R = \ln(y+1)+2$

Answer: $49.9\,\pi\,(\ln(y+1)+2)^2\,\Delta y$. (lbs)

(b) (3 points) Set up an integral equal to the work, in ft-lbs, required to pump all the oil currently in the container to the top of the container. Do not simplify or evaluate the integral.

Work per layer: force as in (a) times dist. from y to 4 i.e. $(4-y)$

$\approx (49.9\pi(\ln(y+1)+2)^2\Delta y)(4-y)$

"Adding up" for $0 \le y \le 2$:

Answer: $\int_0^2 49.9\pi\,(\ln(y+1)+2)^2(4-y)\,dy$

(c) (3 points) Suppose the container already contains 2 feet of paraffin oil, as in the picture above. You wish to pump more oil into the container, from the ground level (x-axis) into the remaining empty space in the container, to fill it up. Set up an integral equal to the work necessary to completely fill the empty part of the container. Do not simplify or evaluate the integral.

calculate work per each layer in top 2 ft i.e. $\boxed{2 \le y \le 4}$ $\leftarrow$ bounds

Weight is like in (a)

distance to be moved is: from x-axis $(y=0)$ to y $\Rightarrow$ dist $= y$

Answer: $\int_2^4 49.9\pi\,(\ln(y+1)+2)^2\,y\,dy$

8. (10 points) Find the solution of the differential equation that satisfies the given initial condition:

$$\frac{dy}{dx} = \frac{xy^3}{\sqrt{3+x^2}}, \qquad y(1) = 1.$$

For full credit, show all steps and write your answer in explicit form, $y = f(x)$.

Separate the variables & integrate:

$$\int \frac{1}{y^3}\,dy = \int \frac{x}{\sqrt{3+x^2}}\,dx$$

$$\frac{y^{-2}}{-2} = \int \frac{1}{\cancel{x}}\,\cancel{x}\,du.$$

$u = \sqrt{3+x^2}$
$u^2 = 3+x^2$
$\cancel{2}u\,du = \cancel{2}x\,dx$

$$-\frac{1}{2y^2} = \sqrt{3+x^2} + c$$

$$\frac{1}{y^2} = -2\sqrt{3+x^2} + c_1 \qquad c_1 = -2c$$

$y(1)=1$: $\quad 1 = -2\sqrt{4} + c_1 \Rightarrow c_1 = 5$

$$\frac{1}{y^2} = 5 - 2\sqrt{3+x^2}$$

$$y^2 = \frac{1}{5-2\sqrt{3+x^2}}$$

$$y = \pm\frac{1}{\sqrt{5-2\sqrt{3+x^2}}}$$

Since $y(1)=1>0 \Rightarrow y = \dfrac{1}{\sqrt{5-2\sqrt{3+x^2}}} = \left(5-2\sqrt{3+x^2}\right)^{-1/2}$

(positive one)

Answer: $y = \dfrac{1}{\sqrt{5-2\sqrt{x^2+3}}}$

9. A rumor is spreading through a college with a population of 2,000 students. The rumor is spreading at a rate proportional to the product of the number of students who have heard it and the number of students who have not heard it. Ten days ago, 50 students had heard the rumor. Today, 120 students have.

Let $P(t)$ denote the number of students who have heard the rumor, where t denotes the number of days since ten days ago (i.e. ten days ago corresponds to $t = 0$.)

(a) (4 points) Set up a differential equation for P, modeling the spread of rumor. Also list all the initial conditions for $P(t)$.

Differential equation: $dP/dt = k P(2000 - P)$.

Initial conditions: $P(0) = 50, \quad P(10) = 120$

(b) (10 points) Solve the differential equation and compute how many students will have heard the rumor 10 days from now.

$$\int \frac{1}{P(2000-P)}\, dP = \int k\, dt$$ by Partial Fractions. (details omitted)

$$\int \frac{1/2000}{P} + \frac{1/2000}{2000-P}\, dP = kt + C$$

$$\frac{1}{2000} \ln|P| - \frac{1}{2000} \ln|2000-P| = kt + C$$

$$\ln\left|\frac{P}{2000-P}\right| = 2000kt + C_1$$

$$\frac{P}{2000-P} = A e^{2000kt} \qquad (A = \pm e^{C_1})$$

$y(0) = 50 \Rightarrow \frac{5\cancel{0}}{195\cancel{0}} = A \Rightarrow A = \frac{1}{39}$

$y(10) = 120 \Rightarrow \frac{12\cancel{0}}{188\cancel{0}} = \frac{1}{39} e^{2000k(10)} \Rightarrow e^{20{,}000k} = \frac{468}{188}$

$\Rightarrow 20{,}000k = \ln\left(\frac{468}{188}\right) = \ln\left(\frac{117}{47}\right) \Rightarrow k = \frac{1}{20{,}000} \ln\left(\frac{117}{47}\right)$

Alternatively, we can find $P = P(t)$ first, then evaluate

$$P = (2000 - P) A e^{2000kt}$$

$$P(1 + A e^{2000kt}) = 2000 A e^{2000kt}$$

$$\boxed{P = \frac{2000 A e^{2000kt}}{1 + A e^{2000kt}}} \leftarrow P(t)$$

$$= \frac{2000(1/39) e^{0.1 \ln(117/47)t}}{1 + (1/39) e^{0.1 \ln(117/47)t}}$$

$$\boxed{P = \frac{2000 e^{0.1 \ln(117/47)t}}{39 + e^{0.1 \ln(117/47)t}}}$$

Now evaluate at $t = 20$ to get $P(20) \simeq 274$

10 days from now is $t = 20$: $\frac{P}{2000-P} = \frac{1}{39} e^{2000 \frac{1}{20000} \ln\left(\frac{117}{47}\right)(20)}$ ← [here P means $P(20)$ i.e. the population at $t=20$.]

$$\Rightarrow \frac{P}{2000-P} = \frac{1}{39} e^{2\ln(117/47)} \simeq 0.158895\ldots$$

$$P = (2000 - P)(0.158895\ldots)$$

$$P(1 + 0.158895\ldots) = 2000(0.158895\ldots)$$

$$P = \frac{2000(0.158895\ldots)}{1.158895\ldots} = 274.218\ldots$$

Answer: 274 students

(round your final answer to the nearest whole number)